

# Supersymmetric gauge theories, quantization of $\mathcal{M}_{\text{flat}}$ , and conformal field theory

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## Abstract

We review the relations between  $\mathcal{N} = 2$ -supersymmetric gauge theories, Liouville theory and the quantization of moduli spaces of flat connections on Riemann surfaces.

## 1. Introduction

Alday, Gaiotto and Tachikawa [AGT] discovered remarkable relations between the instanton partition functions of certain four-dimensional  $\mathcal{N} = 2$ -supersymmetric gauge theories and the conformal field theory called Liouville theory. These relations will be referred to as the AGT-correspondence. We will discuss an explanation for the AGT-correspondence based on the observation that both instanton partition functions and Liouville conformal blocks are naturally related to certain wave-functions in the quantum theory obtained by quantising the moduli spaces of flat  $\text{PSL}(2, \mathbb{R})$ -connections on certain Riemann surfaces  $C$ . We will be considering a class of gauge theories referred to as class  $\mathcal{S}$ , see [V:7, GMN2] or the contribution [V:1] in this volume. The gauge theories  $\mathcal{G}_{C,\mathfrak{g}}$  have elements labelled by the choice of a Riemann surface  $C$  and a Lie-algebra  $\mathfrak{g}$  of type  $A$ ,  $D$  or  $E$ . In the following we will restrict attention to the case where  $\mathfrak{g} = A_1$ , and denote the corresponding gauge theories as  $\mathcal{G}_C$ . However, the reader will notice that many of the arguments below generalise easily to more general theories of class  $\mathcal{S}$ .

The root for the relations between the gauge theories and moduli spaces of flat connections will be found in the identification of the algebra generated by the supersymmetric Wilson- and 't Hooft loop operators with the algebra of trace-functions which represent natural coordinates for the moduli spaces of flat connections. This algebra may become non-commutative if the gauge theories are defined on curved spaces, or deformed by supersymmetry-preserving deformations like the Omega-deformation [N]. It turns out that the resulting non-commutativity is the same as the one resulting from the quantisation of the relevant moduli spaces of flat connections.

Concerning the other side of the coin we are going to review the definition of the conformal

blocks of Liouville theory. Formulated in the right way, part of the relation to the quantization of moduli spaces of flat connections becomes obvious. There furthermore exists a natural representation of the quantized algebra of trace functions on the spaces of conformal blocks.

We are going to explain how the AGT-correspondence follows from the relation between supersymmetric loop operators and trace functions, combined with certain consequences of unbroken supersymmetry. Knowing precisely which algebra is generated by the supersymmetric loop operators, one may reconstruct expectation values of loop operators on backgrounds like the four-ellipsoid. From these data one may in particular recover the low-energy effective actions of the considered gauge theories. This approach relates the AGT-correspondence to some of the work of Gaiotto, Moore and Neitzke [GMN2, GMN3]. It is in some respects similar to the one used by Nekrasov, Rosly and Shatashvili [NRS] to study the case with Omega-deformation preserving two-dimensional  $\mathcal{N} = 2$  super-Poincaré invariance.

## 2. Theories of class $\mathcal{S}$

### 2.1 $A_1$ theories of class $\mathcal{S}$ .

To a Riemann surface  $C$  of genus  $g$  and  $n$  punctures one may associate [G09, GMN2, V:1] a four-dimensional gauge theory  $\mathcal{G}_C$  with  $\mathcal{N} = 2$  supersymmetry, gauge group  $(\mathrm{SU}(2))^h$ ,  $h := 3g - 3 + n$  and flavor symmetry  $(\mathrm{SU}(2))^n$ . The theories in this class are UV-finite, and therefore characterised by a collection of gauge coupling *constants*  $g_1, \dots, g_h$ . In the cases where  $(g, n) = (0, 4)$  and  $(g, n) = (1, 1)$  one would get the supersymmetric gauge theories commonly referred to as  $N_f = 4$  and  $\mathcal{N} = 2^*$ -theory, respectively. The correspondence between data associated to the surface  $C$  and the gauge theory  $\mathcal{G}_C$  is summarised in the table below.

Riemann surface $C$	Gauge theory $\mathcal{G}_C$
Pants decomposition $\mathcal{C}$ + trivalent graph $\Gamma$ on $C$ , $\sigma = (\mathcal{C}, \Gamma)$	Lagrangian description with action functional $S_\tau^\sigma$
Gluing parameters $q_r = e^{2\pi i \tau_r}$ , $r = 1, \dots, 3g - 3 + n$	UV-couplings $\tau = (\tau_1, \dots, \tau_h)$ , $\tau_r = \frac{4\pi i}{g_r^2} + \frac{\theta_r}{2\pi}$
$r$ -th tube $n$ boundaries	$r$ -th vector multiplet $(A_{r,\mu}, \phi_r, \dots)$ $n$ hypermultiplets
Change of pants decomposition	S-duality

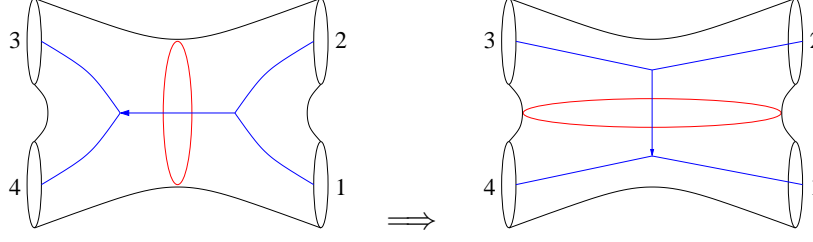


Figure 1: The F-move

More details can be found in [V:1] and references therein. To the  $k$ -th boundary there corresponds a flavor group  $SU(2)_k$  with mass parameter  $M_k$ . The hypermultiplet masses are linear combinations of the parameters  $m_k$ ,  $k = 1, \dots, n$  as explained in more detail in [G09, AGT]. The relevant definitions and results from Riemann surface theory are collected in Appendix A. It is necessary to refine the pants decomposition by introducing the trivalent graph  $\Gamma$  in order to have data that distinguish action functionals with theta angles  $\theta_r$  differing by multiples of  $2\pi$ . This will be done such that

$$S_{\tau+e_r}^\sigma = S_{\tau}^{\delta_r \cdot \sigma}, \quad (2.1)$$

where  $e_r$  is the unit vector with  $r$ -th component equal to one, and  $\delta_r \cdot \sigma$  denotes the action of the Dehn twist along the  $r$ -th tube on  $\sigma = (C, \Gamma)$ , which will map the graph  $\Gamma$  on  $C$  to another one.

## 2.2 Realisation of S-duality

Different Lagrangian descriptions of the theories  $\mathcal{G}_C$  are related by S-duality. Two actions  $S_{\tau_1}^{\sigma_1}$  and  $S_{\tau_2}^{\sigma_2}$  describe different perturbative expansions for one and the same theory. The respective perturbative expansions will be valid in the regimes where all coupling constants  $g_{1,r}$  and  $g_{2,r}$  are small. To formulate the meaning of S-duality more precisely let us assume that there exists a non-perturbative definition of  $\mathcal{G}_C$  allowing us to define normalised expectation values of observables  $\mathcal{O}$  like  $\langle\langle \mathcal{O} \rangle\rangle_{\mathcal{G}_{C_\tau}}$  non-perturbatively as functions of  $\tau$ , a set of parameters for the complex structure on  $C$ . S-duality holds if for each observable  $\mathcal{O}$  there exist functionals  $\mathcal{F}_{\mathcal{O}}^{\sigma_i}$  constructed using the fields in actions  $S_{\tau_i}^{\sigma_i}$  together with choices of coupling constants  $\tau_i = \tau_i(\tau)$ ,  $i = 1, 2$ , such that

$$\langle\langle \mathcal{O} \rangle\rangle_{\mathcal{G}_{C_\tau}} \asymp \langle\langle \mathcal{F}_{\mathcal{O}}^{\sigma_1} \rangle\rangle_{S_{\tau_1}^{\sigma_1}} \quad \text{and} \quad \langle\langle \mathcal{O} \rangle\rangle_{\mathcal{G}_{C_\tau}} \asymp \langle\langle \mathcal{F}_{\mathcal{O}}^{\sigma_2} \rangle\rangle_{S_{\tau_2}^{\sigma_2}}, \quad (2.2)$$

in the sense of equality of asymptotic expansions.

The passage from one Lagrangian description  $S_\tau^\sigma$  to another may be decomposed into the elementary S-duality transformations corresponding to the cases where one of the coupling constants  $g_r$  gets large, while all others  $g_s$ ,  $s \neq r$  stay small. The arguments given in [G09] suggest that S-duality is realized in the following way: In the regime where  $q_r = e^{2\pi i \tau_r} \rightarrow 1$  one may use the Lagrangian description with action  $S_{\tau'}^{\sigma'}$  associated to the data  $\sigma'_{;r} = (C_{;r}, \Gamma_{;r})$  obtained from

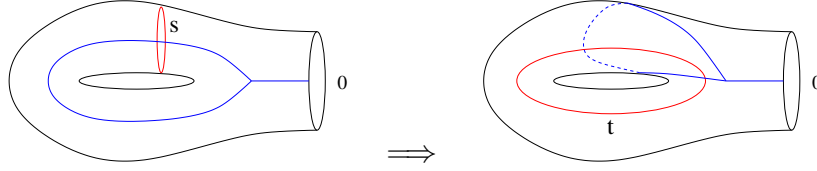


Figure 2: The S-move

$\sigma = (\mathcal{C}, \Gamma)$  by a local modification which is defined as follows: There is a unique subsurface  $C_r \hookrightarrow C$  isomorphic to either  $C_{0,4}$  or  $C_{1,1}$  that contains  $\gamma_r$  in the interior of  $C_r$ .  $\sigma_{;r} = (\mathcal{C}_{;r}, \Gamma_{;r})$  is defined by local substitutions within  $C_r$  depicted in Figures 1 and 2 for the two cases, respectively. If  $C_r = C_{0,4}$  there is another strongly coupled regime which can be described in terms of a dual action. It corresponds to  $q_r \rightarrow \infty$ , and the dual action  $S_{\tau'}^{\sigma_{;r}}$  is associated to the data  $\sigma_{;r}$  obtained from  $\sigma$  by the composition of the B-move depicted in Figure 3 with an F-move.

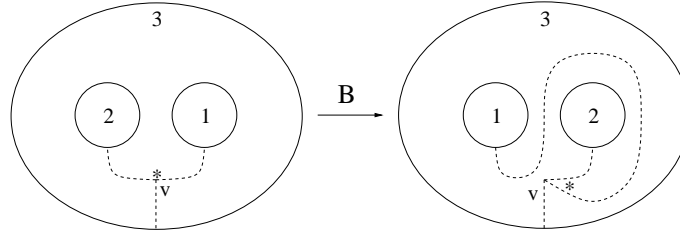


Figure 3: The B-move, represented by (indecently) looking into the pair of pants from above.

An important feature of the mapping between the respective sets of observables is that the Wilson- and 't Hooft loops defined using  $S_{\tau}^{\sigma}$  will correspond to the 't Hooft and Wilson loops defined using  $S_{\tau'}^{\sigma_{;r}}$ , respectively. This is the main feature we shall use in the following.

Any transition between two pants decompositions  $\sigma_1$  and  $\sigma_2$  can be decomposed into the elementary F-, S-, and B-moves. It follows that the groupoid of S-duality transformations coincides with the Moore-Seiberg groupoid for the gauge theories of class  $\mathcal{S}$ , see Appendix A.2.

### 2.3 Gauge theories $\mathcal{G}_C$ on ellipsoids

It may be extremely useful to study quantum field theories on compact Euclidean space-times or on compact spaces rather than  $\mathbb{R}^4$ . Physical quantities get finite size corrections which encode deep information on the quantum field theory we study. The zero modes of the fields become dynamical, and have to be treated quantum-mechanically.

In the case of supersymmetric quantum field theories there are not many compact background space-times that allow us to preserve part of the supersymmetry. A particularly interesting family of examples was studied in [HH], extending the seminal work of Pestun [Pe]. A review can be found in the Article [V:5] in this volume.

Let us consider gauge theories  $\mathcal{G}_C$  on the four-dimensional ellipsoid

$$E_{\epsilon_1, \epsilon_2}^4 := \{ (x_0, \dots, x_4) \mid x_0^2 + \epsilon_1^2(x_1^2 + x_2^2) + \epsilon_2^2(x_3^2 + x_4^2) = 1 \}. \quad (2.3)$$

It was shown in [Pe, HH], see also [V:5], for some examples of gauge theories  $\mathcal{G}_C$  that one of the supersymmetries  $Q$  is preserved on  $E_{\epsilon_1, \epsilon_2}^4$ . It should be possible to generalize the proof of existence of an unbroken supersymmetry  $Q$  to all four-dimensional  $\mathcal{N} = 2$  supersymmetric field theories with a Lagrangian description.

Interesting physical quantities include the partition function  $\mathcal{Z}_{\mathcal{G}_C}$ , or more generally expectation values of supersymmetric loop operators  $\mathcal{L}_\gamma$  such as the Wilson- and 't Hooft loops. Such quantities are formally defined by the path integral over all fields on  $E_{\epsilon_1, \epsilon_2}^4$ . It was shown in a few examples for gauge theories from class  $\mathcal{S}$  in [Pe, HH], reviewed in [V:5], how to evaluate this path integral by means of the localization technique. A variant of the localization argument was used to show that the integral over all fields actually reduces to an integral over the locus in field space where the scalars  $\phi_r$  take constant *real* values  $\phi_r = \text{diag}(a_r, -a_r) = \text{const}$ , and all other fields vanish. This immediately implies that the path integral reduces to an ordinary integral over the variables  $a_r$ . It seems clear that this argument can be generalized to all theories of class  $\mathcal{S}$  with a Lagrangian.

For some theories  $\mathcal{G}_C$  it was found in [Pe] that the result of the localization calculation of the partition function takes the form

$$Z_{E_{\epsilon_1, \epsilon_2}^4}^{\mathcal{G}_C}(m, \tau; \epsilon_1, \epsilon_2) = \int d\mu(a) |\mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2)|^2. \quad (2.4)$$

The main ingredients are the instanton partition function  $\mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2)$  which depends on the zero modes  $a = (a_1, \dots, a_h)$  of the scalar fields, hypermultiplet mass parameters  $m = (m_1, \dots, m_n)$ , UV gauge coupling constants  $\tau = (\tau_1, \dots, \tau_h)$ , and two parameters  $\epsilon_1, \epsilon_2$ . The instanton partition functions can be defined as the partition function of the Omega-deformation of  $\mathcal{G}_C$  on  $\mathbb{R}^4$  [N], and may be calculated by means of the instanton calculus [LNS, MNS1, MNS2, NS04], as reviewed in [V:3] in this volume.

It is expected that the form (2.4) will hold for arbitrary theories  $\mathcal{G}_C$ , but the instanton partition function  $\mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2)$  can only be calculated for the cases where  $C$  has genus 0 or 1, and the pants decomposition is of linear or circular quiver type, respectively.

## 2.4 Supersymmetric loop operators

Supersymmetric Wilson loops can be defined as path-ordered exponentials of the general form

$$W_{r,i} := \text{Tr } \mathcal{P} \exp \left[ \oint_C ds (i\dot{x}^\mu A_\mu^r + |\dot{x}| \phi^r) \right]. \quad (2.5a)$$

The choice of contour  $\mathcal{C}$  is severely constrained by the requirement that the resulting observable is supersymmetric. Two possible choices for the four-manifold  $M^4$  of interest are  $M^4 = \mathbb{R}^3 \times S^1$  and the four-ellipsoid. In the first case one may take a contour  $\mathcal{C}$  that wraps the  $S^1$ . For the case  $M^4 = E_{\epsilon_1 \epsilon_2}^4$  it was shown in [Pe, HH, GOP] that these observables are left invariant by the supersymmetry  $Q$  preserved on  $E_{\epsilon_1, \epsilon_2}^4$  if  $\mathcal{C}$  is one of the contours  $\mathcal{C}_i$ ,  $i = 1, 2$ , with  $\mathcal{C}_1$  and  $\mathcal{C}_2$  being the circles with constant  $(x_0, x_3, x_4) = 0$  and  $(x_0, x_1, x_2) = 0$ , respectively. Throughout this section we will assume that  $\mathcal{C}$  is identified with one of the two  $\mathcal{C}_i$ .

The 't Hooft loop observables  $T_{r,i}$ ,  $i = 1, 2$ , can be defined semiclassically for vanishing theta-angles  $\theta = 0$  by the boundary condition

$$F_r \sim \frac{B_r}{4} \epsilon_{klm} \frac{x^k}{|\vec{x}|^3} dx^m \wedge dx^l, \quad (2.6)$$

near the contour  $\mathcal{C}$ . The coordinates  $x^k$ ,  $k = 1, 2, 3$ , are local coordinates for the space transverse to  $\mathcal{C}_i$ , and  $B$  is an element of the Cartan subalgebra of  $SU(2)$ . In order to get supersymmetric observables one needs to have a corresponding singularity at  $S_i^1$  for the scalar fields  $\phi_r$ . For the details of the definition and the generalization to  $\theta \neq 0$  we refer to [GOP].

Application of the localisation technique to the calculation of Wilson loop operators [Pe, HH], see [V:5, V:6] for reviews, leads to results of the form

$$\langle W_{r,i} \rangle_{E_{\epsilon_1 \epsilon_2}^4} = \int d\mu(a) |\mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2)|^2 2 \cosh(2\pi a_r / \epsilon_i), \quad (2.7)$$

where  $i = 1, 2$ . A rather nontrivial extension of the method from [Pe] allows one to treat the case of 't Hooft loops [GOP] as well, see [V:6] for a review. The result is of the following form:

$$\langle T_{r,i} \rangle_{E_{\epsilon_1, \epsilon_2}^4} = \int d\mu(a) (\mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2))^* \mathcal{D}_{r,i} \mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2), \quad (2.8)$$

with  $\mathcal{D}_{r,i}$  being a difference operator acting only on the variable  $a_r$  of  $\mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2)$ , which has coefficients that depend on  $a, m$  and  $\epsilon_i$ , in general.

## 2.5 Relation to quantum Liouville theory

The authors of [AGT] observed in some examples of theories from class  $\mathcal{S}$  that one has (up to inessential factors  $\mathcal{Z}^{\text{spur}}(m, \tau; \epsilon_1, \epsilon_2)$ ) an equality between the instanton partition functions and the conformal blocks  $\mathcal{Z}^{\text{Liou}}(\beta, \alpha, \tau; b)$  of Liouville theory,

$$\mathcal{Z}^{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2) = \mathcal{Z}^{\text{spur}}(m, \tau; \epsilon_1, \epsilon_2) \mathcal{Z}^{\text{Liou}}(\beta, \alpha, q; b), \quad (2.9)$$

assuming a suitable dictionary between the variables involved. The "spurious" factor  $\mathcal{Z}^{\text{spur}}(m, \tau; \epsilon_1, \epsilon_2)$  will turn out to be inessential, dropping out of normalised expectation values

$$\langle\langle \mathcal{L}_\gamma \rangle\rangle_{E_{\epsilon_1, \epsilon_2}^4} := (\langle 1 \rangle_{E_{\epsilon_1, \epsilon_2}^4})^{-1} \langle \mathcal{L}_\gamma \rangle_{E_{\epsilon_1, \epsilon_2}^4}, \quad (2.10)$$

as follows easily from the general form of the results for the expectation values quoted in (3.7), and is therefore called “spurious”.

We’ll now briefly review the definition of the right hand side of (2.9) for the cases of Riemann surfaces  $C$  of genus zero with  $n$  punctures. The definition for Riemann surfaces  $C$  of arbitrary genus is discussed in [TV13].

The Virasoro algebra  $\text{Vir}_c$  has generators  $L_n$ ,  $n \in \mathbb{Z}$ , and relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}. \quad (2.11)$$

The relevant conformal blocks can be constructed using chiral vertex operators. Let us use the notation  $\Delta_\alpha := \alpha(Q - \alpha)$ , with  $Q$  being a variable parameterising the value  $c$  of the central element in (2.11) as  $c = 1 + 6Q^2$ . We will denote the highest weight representation with weight  $\Delta_\beta$  by  $\mathcal{V}_\beta$ . A chiral vertex operator is an operator  $V_{\beta_2\beta_1}^\alpha(z) : \mathcal{V}_{\beta_1} \rightarrow \mathcal{V}_{\beta_2}$  that satisfies the crucial intertwining property

$$[L_n, V_{\beta_2\beta_1}^\alpha(z)] = z^n(z\partial_z + (n+1)\Delta_\alpha)V_{\beta_2\beta_1}^\alpha(z). \quad (2.12)$$

The property (2.12) defines the operator  $V_{\beta_2\beta_1}^\alpha(z)$  as a formal power series in  $z^k$  uniquely up to multiplication with a complex number. The normalization freedom can be parameterized by the number  $N_{\beta_2\beta_1}^\alpha$  defined by

$$V_{\beta_2\beta_1}^\alpha(z) e_{\beta_1} = z^{\Delta_{\beta_2} - \Delta_{\beta_1} - \Delta_\alpha} [N_{\beta_2\beta_1}^\alpha e_{\beta_2} + \mathcal{O}(z)], \quad (2.13)$$

where  $e_\beta$  is the highest weight vector of the representation  $\mathcal{V}_\beta$ . A particularly useful choice for the normalization factor  $N_{\beta_2\beta_1}^\alpha$  will be

$$N_{\beta_2\beta_1}^\alpha = \sqrt{C(\bar{\alpha}_3, \alpha_2, \alpha_1)}, \quad (2.14)$$

where  $\bar{\alpha}_3 = Q - \alpha_3$ , and  $C(\alpha_3, \alpha_2, \alpha_1)$  is the three-point function in Liouville theory. An explicit formula for  $C(\alpha_3, \alpha_2, \alpha_1)$  was conjectured in [DO, ZZ], and a derivation was subsequently presented in [T01].

Using the invariant bilinear form  $\langle \cdot, \cdot \rangle_\beta : \mathcal{V}_\beta \otimes \mathcal{V}_\beta \rightarrow \mathbb{C}$  one may then construct conformal blocks as matrix elements of products of chiral vertex operators such as

$$\mathcal{Z}_s^{\text{Liou}}(\beta, \alpha, q; b) := \langle e_{\alpha_n}, V_{\alpha_n, \beta_{n-3}}^{\alpha_{n-1}}(z_{n-1}) V_{\beta_{n-3}, \beta_{n-2}}^{\alpha_{n-2}}(z_{n-2}) \cdots V_{\beta_1 \alpha_1}^{\alpha_2}(z_2) e_{\alpha_1} \rangle_{\alpha_n}. \quad (2.15)$$

The parameters  $q = (q_1, \dots, q_{n-3})$  are given by the ratios  $q_r = z_{r+1}/z_{r+2}$ , with  $r = 1, \dots, n-3$ . Equation (2.15) defines conformal blocks associated to particular pants decompositions of  $C_{0,n}$ . In the case of  $n = 4$ , for example, one gets the conformal blocks associated to the pants decomposition depicted on the left of Figure 1.

We may now state the dictionary between the variables appearing in the relation (2.9) between Liouville conformal blocks and the instanton partition functions of the corresponding gauge theories:

$$q_r = \frac{z_{r+1}}{z_{r+2}} = e^{2\pi i \tau_r}, \quad \beta_r = \frac{Q}{2} + i \frac{a_r}{\hbar}, \quad r = 1, \dots, n-3, \quad (2.16a)$$

$$\alpha_k = \frac{Q}{2} + i \frac{M_k}{\hbar}, \quad k = 1, \dots, n, \quad \hbar^2 = \epsilon_1 \epsilon_2. \quad (2.16b)$$

In order to construct conformal blocks associated to general pants decompositions of surfaces  $C_{0,n}$  of genus zero let us introduce the descendants of a chiral vertex operator  $V_{\beta_2\beta_1}^\alpha(z)$ . The descendants may be defined as the family of operators  $V_{\beta_2\beta_1}^\alpha[v](z) : \mathcal{V}_{\beta_1} \rightarrow \mathcal{V}_{\beta_2}$  that satisfy

$$\begin{aligned} V_{\beta_2\beta_1}^\alpha[L_{-2}v](z) &= : T(z) V_{\beta_2\beta_1}^\alpha[v](z) :, & V_{\beta_2\beta_1}^\alpha[e_\alpha](z) &= V_{\beta_2\beta_1}^\alpha(z), \\ V_{\beta_2\beta_1}^\alpha[L_{-1}v](z) &= \partial_z V_{\beta_2\beta_1}^\alpha[v](z), \end{aligned} \quad (2.17)$$

where  $: T(z) V_{\beta_2\beta_1}^\alpha[v](z) :$  is defined as

$$: T(z) V_{\beta_2\beta_1}^\alpha[v](z) : = \sum_{n \leq -2} z^{-n-2} L_n V_{\beta_2\beta_1}^\alpha[v](z) + V_{\beta_2\beta_1}^\alpha[v](z) \sum_{n \geq -1} z^{-n-2} L_n. \quad (2.18)$$

With the help of the descendants one has a new way to compose chiral vertex operators, allowing us, for example, to construct conformal blocks on  $C = \mathbb{P}^1 \setminus \{0, z_2, z_3, \infty\}$  as

$$\mathcal{Z}_t^{\text{Liou}}(\beta, \alpha, z; b) := \langle e_{\alpha_4}, V_{\alpha_4\alpha_1}^\beta [V_{\beta\alpha_2}^{\alpha_3}(z_3 - z_2) e_{\alpha_2}](z_2) e_{\alpha_1} \rangle_{\alpha_4}. \quad (2.19)$$

This conformal block is associated to the pants decomposition on the right of Figure 1. By considering arbitrary compositions of chiral vertex operators one may construct conformal blocks associated to arbitrary pants decompositions of a surface  $C$  with genus zero and  $n$  boundaries.

The relations (2.9) have fully been proven [AFLT] in the cases where the relevant conformal blocks are of the form (2.15) corresponding to the so-called linear quiver gauge theories. It is not straightforward to generalise this proof to more general pants decompositions like those corresponding to conformal blocks of the form (2.19). The technical difficulties encountered for more general pants decompositions are considerable and not yet resolved in general, see [HKS] for partial results in this direction.

### 3. Reduction to quantum mechanics

#### 3.1 Localization as reduction to zero mode quantum mechanics

We may assign to the expectation values  $\langle \mathcal{L} \rangle$  of a loop observable  $\mathcal{L}$  an interpretation in terms of expectation values of operators  $L_{\mathcal{L}}$  which act on the Hilbert space obtained by canonical



quantization of the gauge theory  $\mathcal{G}_C$  on the space-time  $\mathbb{R} \times E_{\epsilon_1, \epsilon_2}^3$ , where  $E_{\epsilon_1, \epsilon_2}^3$  is the three-dimensional ellipsoid defined as

$$E_{\epsilon_1, \epsilon_2}^3 := \{ (x_1, \dots, x_4) \mid \epsilon_1^2(x_1^2 + x_2^2) + \epsilon_2^2(x_3^2 + x_4^2) = 1 \}. \quad (3.1)$$

This is done by interpreting the coordinate  $x_0$  for  $E_{\epsilon_1, \epsilon_2}^4$  as Euclidean time. Noting that  $E_{\epsilon_1, \epsilon_2}^4$  looks near  $x_0 = 0$  as  $\mathbb{R} \times E_{\epsilon_1, \epsilon_2}^3$ , we expect to be able to represent partition functions  $\mathcal{Z}_{\mathcal{G}_C}(E_{\epsilon_1, \epsilon_2}^4)$  or expectation values  $\langle \mathcal{L} \rangle_{\mathcal{G}_C(E_{\epsilon_1, \epsilon_2}^4)}$  as matrix elements of states in the Hilbert space  $\mathcal{H}_{\mathcal{G}_C}$  defined by canonical quantization of  $\mathcal{G}_C$  on  $\mathbb{R} \times E_{\epsilon_1, \epsilon_2}^3$ . More precisely

$$\mathcal{Z}_{\mathcal{G}_C}(E_{\epsilon_1, \epsilon_2}^4) = \langle \tau \mid \tau \rangle, \quad \langle \mathcal{L} \rangle_{E_{\epsilon_1, \epsilon_2}^4} = \langle \tau \mid \mathcal{L}_\mathcal{L} \mid \tau \rangle, \quad (3.2)$$

where  $\langle \tau \mid$  and  $\mid \tau \rangle$  are the states created by performing the path integral over the upper/lower half-ellipsoid

$$E_{\epsilon_1, \epsilon_2}^{4, \pm} := \{ (x_0, \dots, x_4) \mid x_0^2 + \epsilon_1^2(x_1^2 + x_2^2) + \epsilon_2^2(x_3^2 + x_4^2) = 1, \pm x_0 > 0 \}, \quad (3.3)$$

respectively, and  $\mathcal{L}_\mathcal{L}$  is the operator that represents the observable  $\mathcal{L}$  within  $\mathcal{H}_{\mathcal{G}_C}$ .

The form (2.7), (2.8) of the loop operator expectation values is naturally interpreted in the Hamiltonian framework as follows. In the functional Schroedinger picture one would represent the expectation values  $\langle \mathcal{L} \rangle_{E_{\epsilon_1, \epsilon_2}^4}$  schematically in the following form

$$\langle \mathcal{L} \rangle_{E_{\epsilon_1, \epsilon_2}^4} = \int [\mathcal{D}\Phi] (\Psi[\Phi])^* \mathcal{L}_\mathcal{L} \Psi[\Phi], \quad (3.4)$$

the integral being extended over all field configuration on the three-ellipsoid  $E_{\epsilon_1, \epsilon_2}^3$  at  $x_0 = 0$ . The wave-functional  $\Psi[\Phi]$  is defined by means of the path integral over the lower half-ellipsoid  $E_{\epsilon_1, \epsilon_2}^{4, -}$  with Dirichlet-type boundary conditions defined by the field configuration  $\Phi$ .

The fact that the path integral localizes to the locus  $\text{Loc}_C$  defined by constant values  $\phi_r = \text{diag}(a_r, -a_r) = \text{const.}$  of the scalars and zero values for all other fields [Pe, HH, V:5] implies that the path integral in (3.4) can be reduced to an ordinary integral of the form

$$\langle \mathcal{L} \rangle_{E_{\epsilon_1, \epsilon_2}^4} = \int da (\Psi_\tau(a))^* \pi_0(\mathcal{L}_\mathcal{L}) \Psi_\tau(a), \quad (3.5)$$

with  $\Psi_\tau(a)$  defined by means of the path integral over the lower half-ellipsoid  $E_{\epsilon_1, \epsilon_2}^{4, -}$  with Dirichlet boundary conditions  $\Phi \in \text{Loc}_C$ ,  $\phi_r = \text{diag}(a_r, -a_r)$ ,  $r = 1, \dots, h$ . The Dirichlet boundary condition  $\Phi \in \text{Loc}_C$ ,  $\phi_r = a_r$  is naturally interpreted as defining a Hilbert subspace  $\mathcal{H}_0$  within  $\mathcal{H}_{\mathcal{G}_C}$ . States in  $\mathcal{H}_0$  can, by definition, be represented by wave-functions  $\Psi(a)$ ,  $a = (a_1, \dots, a_h)$ .  $\pi_0(\mathcal{L}_\mathcal{L})$  is the projection of  $\mathcal{L}_\mathcal{L}$  to  $\mathcal{H}_0$ .

Note that the boundary condition  $\Phi \in \text{Loc}_C$  preserves the supercharge  $Q$  used in the localization calculations of [Pe, HH, V:5] – that's just what defined the locus  $\text{Loc}_C$  in the first place. We

may therefore use the arguments from [Pe, HH] to identify the wave-functions  $\Psi_\tau(a)$  in (3.5) with the instanton partition functions,

$$\Psi_\tau(a) = \mathcal{Z}_{\text{inst}}(a, m, \tau; \epsilon_1, \epsilon_2). \quad (3.6)$$

The form of the results for expectation values of loop observables quoted in (2.7), (2.8) is thereby naturally explained.

### 3.2 S-duality of expectation values

In each Lagrangian description with action  $S_\tau^\sigma$  one will be able to express loop operator expectation values in the form

$$\langle \mathcal{L} \rangle_{E_{\epsilon_1, \epsilon_2}^4}^{S_\tau^\sigma} = \int da (\Psi_\tau^\sigma(a))^* \mathcal{D}_\mathcal{L}^\sigma \Psi_\tau^\sigma(a), \quad (3.7)$$

defining representations of the algebra  $\mathcal{A}_{\epsilon_1 \epsilon_2}$  in terms of operators  $\mathcal{D}_\mathcal{L}^\sigma$ . The Wilson loops  $W_{r,1}$  and  $W_{r,2}$  act diagonally as operators of multiplication by  $2 \cosh(2\pi a_r / \epsilon_1)$  and  $2 \cosh(2\pi a_r / \epsilon_2)$ , respectively. The 't Hooft loops  $T_{r,i}$  will be represented by difference operators denoted as  $\mathcal{D}_{r,i}^\sigma$ .

In order for S-duality to hold, we need that the representations of the algebra of loop operators associated to any two pants decompositions  $\sigma_1$  and  $\sigma_2$  are unitarily equivalent. This means in particular that the eigenfunctions  $\Psi_\tau^{\sigma_1}(a)$  and  $\Psi_\tau^{\sigma_2}(a)$  must be related by an integral transformations of the form<sup>1</sup>

$$\Psi_\tau^{\sigma_2}(a_2) = \int da_1 K_{\sigma_2 \sigma_1}(a_2, a_1) \Psi_\tau^{\sigma_1}(a_1). \quad (3.8)$$

If  $S_\tau^\sigma$  and  $S_{\tau'}^{\sigma'}$  are two actions with  $\tau$  and  $\tau'$  differing only by shifts of the theta-angle  $\theta_r \rightarrow \theta_r + 2\pi k_r$ , it follows from (2.1) that we must have

$$\Psi_{\tau'}^{\sigma'}(a) = \Psi_\tau^{\sigma'}(a), \quad (3.9)$$

with  $\tau' = \tau + k_r e_r$ . By using the transformations (3.8) one finds that we must have

$$\Psi_{\mu, \tau}^\sigma(a) = \Psi_\tau^{\mu, \sigma}(a), \quad (3.10)$$

for any Dehn twist  $\mu \in \text{MCG}(C)$ . The notation  $\Psi_{\mu, \tau}^\sigma(a)$  on the left hand side denotes the analytic continuation of  $\Psi_\tau^\sigma(a)$  with respect to  $\tau$  defined by the element  $\mu \in \text{MCG}(C)$ .

By combining (3.8) and (3.10) we get

$$\Psi_{\mu, \tau}^\sigma(a_2) = \int da_1 K_{\mu, \sigma}(a_2, a_1) \Psi_\tau^\sigma(a_1). \quad (3.11)$$

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<sup>1</sup>Considering theories  $\mathcal{G}_C$  associated to Riemann surfaces with genus  $g > 1$  one has to allow for an additional factor on the right hand side of the relation (3.8). This is discussed in [TV13].

Assuming that we know the kernels  $K_{\mu,\sigma,\sigma}(a_2, a_1)$ , we would thereby get a Riemann-Hilbert type problem<sup>2</sup> for the wave-functions  $\Psi_\tau^\sigma(a_1)$ . Equation (3.11) describes the effect of a monodromy in the gauge theory parameter space in terms of an integral transformation with kernel  $K_{\mu,\sigma,\sigma}(a_2, a_1)$ .

Let's note, however, that the kernels  $K_{\mu,\sigma,\sigma}(a_2, a_1)$  are by no means arbitrary: They are strongly constrained by the fact that (3.8) must intertwine the representations of the algebra  $\mathcal{A}_{\epsilon_1\epsilon_2}$  defined by the actions  $S^{\sigma_1}$  and  $S^{\sigma_2}$ , respectively. Concretely, we must have, in particular,

$$\begin{aligned} \overrightarrow{\mathcal{D}}_{r,i}^{\sigma_2} \cdot K_{\sigma_2\sigma_1}(a_2, a_1) &= K_{\sigma_2\sigma_1}(a_2, a_1) 2 \cosh(2\pi a_{1,r}/\epsilon_i), \\ K_{\sigma_2\sigma_1}(a_2, a_1) \cdot \overleftarrow{\mathcal{D}}_{r,i}^{\sigma_1} &= 2 \cosh(2\pi a_{2,r}/\epsilon_i) K_{\sigma_2\sigma_1}(a_2, a_1), \end{aligned} \quad (3.12)$$

expressing the fact that S-duality exchanges Wilson and 't Hooft loops. The equations represent a system of difference equations that turns out to determine  $K_{\sigma_2\sigma_1}(a_2, a_1)$  uniquely up to normalization. This means that the kernels are essentially determined by the representation theory of the algebra  $\mathcal{A}_{\epsilon_1\epsilon_2}$ .

We will in the following describe how to identify the algebra  $\mathcal{A}_{\epsilon_1\epsilon_2}$ . This information may then be used [TV13] to determine the kernels  $K_{\mu,\sigma,\sigma}(a_2, a_1)$  defining the Riemann-Hilbert problem (3.11). Fixing the  $\tau$ -asymptotics by means of perturbative information one gets a Riemann-Hilbert problem which has an essentially unique solution, thereby characterizing the wave-functions  $\Psi_\tau^\sigma(a)$  completely.

Keeping in mind (3.6) we conclude that the instanton partition functions  $\mathcal{Z}^{\text{inst}}$  can be characterized using the representation theory of  $\mathcal{A}_{\epsilon_1\epsilon_2}$ . Note that the prepotential  $\mathcal{F}$  giving the low-energy effective action of  $\mathcal{G}_C$  is recovered from  $\mathcal{Z}^{\text{inst}}$  via  $\mathcal{F} = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_1 \epsilon_2 \log \mathcal{Z}^{\text{inst}}$ . This means that the low-energy effective action is encoded abstractly within the algebra  $\mathcal{A}_{\epsilon_1\epsilon_2}$ . These observations motivate why this algebra was called “non-perturbative skeleton” of  $\mathcal{G}_C$  in [TV13].

## 4. The algebra of loop operators

In order to realise the program outlined at the end of the previous section it will be essential to know the algebra  $\mathcal{A}_{\epsilon_1\epsilon_2}$  precisely. We are now going to explain how  $\mathcal{A}_{\epsilon_1\epsilon_2}$  is related to the non commutative algebra obtained by quantising the space of functions on the moduli space of flat  $\text{PSL}(2)$ -connections. This section is meant to give a guide to the literature on the known

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<sup>2</sup>The Riemann-Hilbert problem is often formulated as the problem to find vectors of multivalued analytic functions on a punctured Riemann surface  $C$  with given monodromy, a representation of  $\pi_1(C)$  in  $SL(N, \mathbb{C})$ . Our equation (3.11) generalises the Riemann-Hilbert problem in two ways: The Riemann surface  $C$  is replaced by the moduli space  $\mathcal{M}(C)$  of complex structures on the surface  $C$ , and the monodromy takes values in the group of unitary transformations of an infinite-dimensional Hilbert-space rather than  $SL(N, \mathbb{C})$ .

relations between the algebra generated by the supersymmetric Wilson- and 't Hooft loop operators on the one hand, and the (quantised) algebra of functions on the moduli space of flat  $\mathrm{PSL}(2)$ -connections on the other hand.

#### 4.1 The algebra of supersymmetric loop operators

The algebra of gauge theory observables contains the supersymmetric Wilson- and 't Hooft loop observables. The product of such loop operators will generate further loop observables supported at the same loop  $\mathcal{C}$ . The generalizations of Wilson- and 't Hooft loop operators  $\mathcal{L}_\gamma$  that are generated in this way describe the effect of inserting heavy “dyonic” probe particles, and can therefore be labelled by pairs  $\gamma = (r, s)$  of electric and magnetic charge vectors, see [DMO], and the article [V:6] for a review. We will be interested in the algebra  $\mathcal{A}$  generated by polynomial functions of the loop operators.

One should note that the labelling of loop operators by charges is based on a given Lagrangian description of the theory. A particularly simple example for the dependence of the underlying Lagrangian is provided by the Witten-effect: Two actions  $S_1$  and  $S_2$  which differ only by a shift of the theta-angle  $\theta_r$  by  $2\pi$  will define the same expectation values after proper identification of the loop operators: A loop observable with charge  $\gamma_1$  defined by  $S_1$  gets identified with the loop observable with charge  $\gamma_2$  defined by  $S_2$  iff the magnetic charges coincide and the electric charges of  $\gamma_1$  and  $\gamma_2$  differ by certain multiples of the magnetic charges. A precise statement for the  $A_1$  theories of class  $\mathcal{S}$  of interest here can be found in [DMO], see also [V:6].

It should also be remarked that the precise specification of a gauge theory of class  $\mathcal{S}$  depends on certain discrete topological data [AST, Ta13] defining in particular the set of allowed charges for the line operators. This phenomenon is related to interesting subtleties showing up when the gauge theory  $\mathcal{G}_C$  is studied on more general four-manifolds, but it is not relevant for what is discussed in this article as we are exclusively dealing with four manifolds having the topology of the four-sphere.

#### 4.2 UV versus IR loop operators

It will be instructive to consider the four-ellipsoid  $E_{\epsilon_1\epsilon_2}^4$  in the limit where  $\epsilon_1 = 0$ . In this case the four-ellipsoid  $E_{\epsilon_1,\epsilon_2}^4$  degenerates into  $E_{\epsilon_2}^2 \times \mathbb{R}^2$ , where

$$E_{\epsilon_2}^2 := \{ (x_0, \dots, x_2) \mid x_0^2 + \epsilon_2^2(x_3^2 + x_4^2) = 1 \}. \quad (4.1)$$

This implies that only the Wilson- and 't Hooft loops wrapped on the remaining circle  $\mathcal{C}_2$  will remain. We may still relate expectation values to matrix elements by choosing  $x_0$  as (Euclidean) time coordinate. Near the “equator”  $x_0 = 0$ , the two-ellipsoid  $E_{\epsilon_2}^2$  looks like  $\mathbb{R} \times S^1$ . One may

expect that studying the gauge theory  $\mathcal{G}_C$  on  $\mathbb{R}^2 \times E_{\epsilon_1}^2$  will allow us make contact with the work of Gaiotto, Moore and Neitzke [GMN1]-[GMN3], who have studied the gauge theories  $\mathcal{G}_C$  on the circle compactification  $\mathbb{R}^3 \times S^1$ . Aspects relevant for us are reviewed in [V:2].

Considering the theory  $\mathcal{G}_C$  on  $\mathbb{R}^3 \times S^1$  at low energies, it was argued in [GMN1] that  $\mathcal{G}_C$  becomes effectively represented by a three-dimensional sigma model with hyperkähler target space  $\mathcal{M}(C)$ . This means in particular that the hyperkähler space  $\mathcal{M}(C)$  represents the moduli space of vacua of  $\mathcal{G}_C$  on  $\mathbb{R}^3 \times S^1$ .

The supersymmetric Wilson- and 't Hooft loops supported on  $S^1$  are called UV line operators in [GMN3, V:2]. Vacuum expectation values of these line operators<sup>3</sup>

$$L_\gamma(m) := \langle \mathcal{L}_\gamma \rangle_m, \quad m \in \mathcal{M}(C), \quad (4.2)$$

represent coordinate functions on the moduli space of vacua of  $\mathcal{G}_C$  on  $\mathbb{R}^3 \times S^1$ . We see that the algebra  $\mathcal{A}$  of UV line operators must coincide with a (sub-)algebra of the algebra of functions on the moduli space of vacua  $\mathcal{M}(C)$ .

Other useful sets of coordinate functions for  $\mathcal{M}(C)$  have been defined in [GMN1] using the effective low-energy description of  $\mathcal{G}_C$ : They are denoted as  $\mathcal{X}_\eta(m)$ , are labelled by the charge lattice  $\Gamma$ , and represent Darboux coordinates for the holomorphic symplectic structure  $\Omega$  on  $\mathcal{M}(C)$ . The functions  $\mathcal{X}_\eta(m)$  have been interpreted in [GMN3] as expectation values of IR line operators describing the effect of the insertion of a heavy dyonic source of charge  $\eta$  into the low-energy effective field theory.

It has been argued in [GMN3], see also [V:2], that the expectation values  $L_\gamma(m)$  can be alternatively computed using the effective IR description of  $\mathcal{G}_C$  on  $\mathbb{R}^3 \times S^1$ , leading to a relation between UV and IR line operators of the following form:

$$L_\gamma(m) = \sum_{\eta \in \Gamma} \bar{\Omega}_{\gamma, \eta} \mathcal{X}_\eta(m). \quad (4.3)$$

The positive-integer coefficients  $\bar{\Omega}_{\gamma, \eta}$  have an interesting physical interpretation as an index counting certain BPS states that exist in the presence of line defects [GMN3].

### 4.3 Relation with moduli spaces of flat connections

The following table summarizes known connections between the moduli spaces  $\mathcal{M}_{\text{flat}}(C)$  of flat  $SL(2)$ -connections<sup>4</sup> on the surfaces  $C$  and the moduli space of vacua  $\mathcal{M}(C)$  on  $\mathbb{R}^3 \times S^1$ :

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<sup>3</sup>Comparing with [V:2] let us note that on  $\mathbb{R}^3 \times S^1$  one may consider families of line operators preserving different supersymmetries, parameterised by a parameter  $\zeta$  in [V:2]. We here focus on the case  $\zeta = 1$  corresponding to the line operators studied on  $E_{\epsilon_1 \epsilon_2}^4$ . Let us furthermore note that the label  $\gamma$  used for UV line operators here is used for IR line operators in [V:2].

<sup>4</sup>This may be  $SL(2, \mathbb{C})$ - or  $SL(2, \mathbb{R})$ -connections depending on the context, as will be discussed later.

Riemann surface $C$	Gauge theory $\mathcal{G}_C$
Moduli space of flat connections $\mathcal{M}_{\text{flat}}(C)$	Moduli space of vacua $\mathcal{M}(C)$ on $\mathbb{R}^3 \times S^1$
Trace functions $L_\gamma$ on $\mathcal{M}_{\text{flat}}(C)$	UV line operators $\mathcal{L}_\gamma$
Fock-Goncharov coordinates	IR line operators $\mathcal{X}_\eta$

We have gathered the relevant definitions and results concerning  $\mathcal{M}_{\text{flat}}(C)$  in Appendix B. The mapping between trace functions  $L_\gamma$  and UV line operators  $\mathcal{L}_\gamma$  is defined by identifying the Dehn-Thurston parameters classifying closed loops on  $C$  (see Subsection B.5 for a short summary) with the charge labels  $\gamma = (r, s)$  of the line operators [DMO, V:6].<sup>5</sup> The definition of the Fock-Goncharov coordinates for  $\mathcal{M}_{\text{flat}}(C)$  is briefly reviewed in Appendix B.3, and the relations to IR line operators are discussed in [GMN2, GMN3].

An argument in favor of the identification between  $\mathcal{M}_{\text{flat}}(C)$  and  $\mathcal{M}(C)$  starts by considering the six-dimensional  $(2, 0)$  theory on  $S^1 \times \mathbb{R}^3 \times C$ . Compactifying first on  $C$  and then on  $S^1$  gives the three-dimensional sigma model with target space  $\mathcal{M}(C)$ , as mentioned above. It may alternatively be obtained from the six-dimensional theory by first compactifying on  $S^1$  followed by compactification on  $C$ . After compactifying on  $S^1$  one would then find the maximally supersymmetric five-dimensional super-Yang-Mills theory on  $\mathbb{R} \times \mathbb{R}^2 \times C$ . Further compactification on  $C$  yields a nonlinear sigma-model with target being  $\mathcal{M}_{\text{Hit}}(C)$ , the moduli space of solutions to Hitchin's self-duality equations using a variant of the argument presented in [BJSV]. More details and references can be found in [GMN2, Section 3.1].  $\mathcal{M}_{\text{Hit}}(C)$  is a hyperkähler space naturally related to  $\mathcal{M}_{\text{flat}}(C)$  in one of its hyperkähler structures [Hi, V:2].

A way to find the identification between particular coordinate functions on  $\mathcal{M}_{\text{flat}}(C)$  and the UV line operators summarised in the table above was described in [GMN3, Section 7].

## 4.4 Quantization

An interesting generalization of the set-up considered in Subsection 4.2 (compactification on  $S^1$ ) is obtained by imposing certain twisted boundary conditions with parameter  $b$  along  $S^1$  [GMN3, IOT]. The resulting deformation, denoted  $\mathbb{R}^3 \times_b S^1$  of the background  $\mathbb{R}^3 \times S^1$  is related

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<sup>5</sup>The set of allowed charges  $\gamma = (r, s)$  in a theory  $\mathcal{G}_C$  is generically smaller than the set of allowed Dehn-Thurston parameters [AST, Ta13]. This subtlety does not affect our discussions: For each allowed Dehn-Thurston parameter there *exists* a choice of the extra discrete data specifying gauge theories  $\mathcal{G}_C$  such that the corresponding UV line operator  $\mathcal{L}_\gamma$  can be defined within  $\mathcal{G}_C$ . Having determined the set of allowed charges in the duality frame corresponding to a particular pants decomposition, one may figure out the allowed charges in any other duality frame by some simple rules.

to the Omega-deformation, and it can be used to model the residual effect of the curvature in the vicinity of the circles  $\mathcal{C}_i$  on  $E_{\epsilon_1, \epsilon_2}^4$  which represent the support of the loop operators [V:6].

It has been argued in [GMN3, IOT], see also [V:6], that the effect of the twisted boundary conditions is to deform the algebra  $\mathcal{A}$  into a non-commutative algebra  $\mathcal{A}_b$ . In the case of the  $A_1$  theories of class  $\mathcal{S}$  it was argued in [GMN3] that the resulting algebra is nothing but the quantized algebra of functions on  $\mathcal{M}_{\text{flat}}(C)$ , denoted  $\text{Fun}_{\hbar}(\mathcal{M}_{\text{flat}}(C))$ , here with  $\hbar = b^2$ . There should in particular exist a deformed version of the relation (4.3) between UV and IR line operators. The left hand side of this relation, the deformed UV line operator, should be independent of the choice of coordinates that appear on the right hand side. As different sets of coordinates  $\mathcal{X}_\eta$  are related by (quantized-) cluster transformations, it will suffice to figure out the quantum analog of (4.3) for particular triangulations. This is what was done in [GMN3, Section 11] for the  $A_1$ -case, leading to the conclusion that the algebra generated by the deformed UV line operators is the quantisation of the algebra of trace functions on  $\mathcal{M}_{\text{flat}}(C)$  that will be described in more detail in the following section.

Highly nontrivial support for this proposal has been given by explicit calculations for some theories of class  $\mathcal{S}$  [IOT, V:6]. A rather different line of arguments leading to the same conclusion was proposed by Nekrasov and Witten in [NW].

#### 4.5 Back to the ellipsoid

As mentioned above, one may expect that the twisted boundary conditions defining  $\mathbb{R}^3 \times_b S^1$  would model the residual effect of the curvature in the vicinity of the curves  $\mathcal{C}_i$  on  $E_{\epsilon_1, \epsilon_2}^4$  [IOT, V:6], at least as far as the algebraic properties of loop operators are concerned. The comparison of the results of localisation calculations on the two spaces ([GOP] for  $S^4$ , and [IOT] for  $\mathbb{R}^3 \times_b S^1$ ) provides highly nontrivial quantitative evidence for this claim. In the case of the four ellipsoid  $E_{\epsilon_1, \epsilon_2}^4$  one thereby expects to get a (twisted) product of two copies of  $\text{Fun}_{\hbar}(\mathcal{M}_{\text{flat}}(C))$  associated to the two circles  $\mathcal{C}_i$  supporting supersymmetric loop observables.

However, there is a crucial difference between the cases of  $\mathbb{R}^3 \times_b S^1$  and  $E_{\epsilon_1, \epsilon_2}^4$ . In the case of  $\mathbb{R}^3 \times_b S^1$  one will generically get complex values for expectation values of loop observables which are functions of the scalar expectation values at infinity, the holonomy of the gauge field around  $S^1$  and the complexified gauge coupling constants  $\tau$ . The precise relation was given in [IOT].

In the case of  $E_{\epsilon_1, \epsilon_2}^4$ , on the contrary, one gets only real numbers larger than 2 for the expectation values of Wilson loops from the localisation calculations of [Pe, HH]. By S-duality this will imply that the 't Hooft loops will define positive self-adjoint operators on  $\mathcal{H}_0$  with the same spectrum. This means that the relevant moduli spaces to consider in this case will not be the

moduli spaces  $\mathcal{M}_{\text{flat}}^{\mathbb{C}}(C)$  of flat  $\text{PSL}(2, \mathbb{C})$ -connections, but rather its real slice  $\mathcal{M}_{\text{flat}}^{\mathbb{R}}(C)$  defined by having real values bounded below by 2 for all trace coordinates.

It is known that  $\mathcal{M}_{\text{flat}}^{\mathbb{R}}(C)$  breaks up into finitely many disconnected components  $\mathcal{M}_{\text{flat}}^{\mathbb{R},d}(C)$ ,  $|d| = 0, \dots, 2g - 2 + n$ , and there exists a distinguished component  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  which has the necessary properties. This component is isomorphic to the Teichmüller spaces of Riemann surfaces [Go88, Hi] and therefore referred to as the Teichmüller component, see Appendix B.2.

The resulting situation is summarised in the table below.

Riemann surface $C$	Gauge theory $\mathcal{G}_C$
Quantised algebras of functions $\text{Fun}_{b^2}(\mathcal{M}_{\text{flat}}(C))$	Algebra $\mathcal{A}_b$ generated by Wilson- and 't Hooft loops on $\mathbb{R}^3 \times_b S^1$
Quantized algebras of functions $\text{Fun}_{b^2}(\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)) \tilde{\times} \text{Fun}_{b^{-2}}(\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C))$	Algebra $\mathcal{A}_{\epsilon_1 \epsilon_2}$ generated by Wilson- and 't Hooft loops on $E_{\epsilon_1, \epsilon_2}^4$

The notation  $\tilde{\times}$  indicates that the representatives of the factors commute only up to a sign, in general.

## 5. Quantization of moduli spaces of flat connections

We now have the input we need to develop the program outlined in Subsection 3.2 - the reconstruction of instanton partition functions from the algebra of loop operators. In the rest of this section we shall briefly describe the quantization of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$ .

### 5.1 Quantization of the Fock-Goncharov coordinates

The simplicity of the Poisson brackets of the Fock-Goncharov coordinates makes part of the quantization quite simple. To each edge  $e$  of a triangulation  $\mathfrak{t}$  of a Riemann surface  $C_{g,n}$  associate a quantum operator  $X_e^{\mathfrak{t}}$  corresponding to the classical phase space function  $\mathcal{X}_e^{\mathfrak{t}}$ . Canonical quantization of the Poisson brackets (B.17) yields an algebra  $\mathcal{B}_{\mathfrak{t}}$  with generators  $X_e^{\mathfrak{t}}$  and relations

$$X_e^{\mathfrak{t}}, X_{e'}^{\mathfrak{t}} = e^{2\pi i b^2 n_{ee'}} X_{e'}^{\mathfrak{t}} X_e^{\mathfrak{t}}, \quad (5.1)$$

where  $n_{ee'}$  is the number of intersections of  $e$  with  $e'$ , counted with a sign.

Note furthermore that the variables  $\mathcal{X}_e$  are positive for the Teichmüller component. The scalar product of the quantum theory should realize the phase space functions  $\mathcal{X}_e$  as *positive* self-



adjoint operators  $X_e^t$ . By choosing a polarization one may define a Schrödinger type representations  $\pi_t$  in terms of multiplication and finite shift operators. It can be realized on suitable dense subspaces of the Hilbert space  $\mathcal{H}_t \simeq L^2(\mathbb{R}^{3g-3+n})$ .

There exists a family of automorphisms which describe the relation between the quantized variables associated to different triangulations [F97, Ka1, CF1]. If triangulation  $t_e$  is obtained from  $t$  by changing only the diagonal in the quadrangle containing  $e$ , we have

$$X_{e'}^{t_e} = \begin{cases} X_{e'}^t \prod_{a=1}^{|n_{e'e}|} (1 + e^{\pi i(2a-1)b^2} (X_e^t)^{-\text{sgn}(n_{e'e})})^{-\text{sgn}(n_{e'e})} & \text{if } e' \neq e, \\ (X_e^t)^{-1} & \text{if } e' = e. \end{cases} \quad (5.2)$$

It follows that the quantum theory of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  has the structure of a quantum cluster algebra [FG2].

It is possible to construct [Ka1] unitary operators  $T_{t_1, t_2}$  that represent the quantum cluster transformations (5.2) in the sense that

$$X_e^{t_2} = T_{t_1 t_2}^{-1} \cdot X_e^{t_1} \cdot T_{t_1 t_2}. \quad (5.3)$$

The operators  $T_{t_2, t_1}$  describe the change of representation when passing from the quantum theory associated to triangulation  $t_1$  to the one associated to  $t_2$ . It follows that the resulting quantum theory does not depend on the choice of a triangulation in an essential way.

*As indicated in Section 4.4, one may interpret the coordinates  $\mathcal{X}_e^\tau$  as expectation values of IR line operators. The formula (5.1) describes the quantum deformation induced by the twisted boundary condition on  $\mathbb{R}^3 \times_b S^1$ , and (5.2) describes the behavior of the IR line operators under (quantum-) wall-crossing [GMN1, GMN3, V:2].*

## 5.2 Quantization of the trace functions

There is a simple algorithm (reviewed in Appendix B.7) for calculating the trace functions in terms of the variables  $\mathcal{X}_e^t$  leading to Laurent polynomials in the variables  $\mathcal{X}_e$  of the form

$$L_\gamma = \sum_{\nu \in \mathbb{F}} C_\gamma^t(\nu) \prod_e (\mathcal{X}_e^t)^{\frac{1}{2}\nu_e}, \quad (5.4)$$

where the summation is taken over a finite set  $\mathbb{F}$  of vectors  $\nu \in \mathbb{Z}^{3g-3+2n}$  with components  $\nu_e$ .

*According to [GMN3, V:2] one may interpret the trace functions as UV line operators. Formula (5.4) thereby becomes identified with (4.3).*

For curves  $\gamma$  having  $C_\gamma^t(\nu) \in \{0, 1\}$  for all  $\nu \in \mathbb{F}$  it has turned out to be sufficient to replace  $(\mathcal{X}_e^t)^{\nu_e}$  in (5.4) by  $\exp(\sum_e \nu_e \log X_e^t)$  in order to define the quantum operator  $L_\gamma^t$  associated to a

classical trace function  $L_\gamma$ . For other triangulations one may define  $L_\gamma^{t'}$  using

$$L_\gamma^{t'} = T_{tt'}^{-1} \cdot L_\gamma^t \cdot T_{tt'} . \quad (5.5)$$

It turns out that this is sufficient to define the operators  $L_\gamma^t$  in general [T05]. It follows from (5.5) that we may regard the algebras of quantised trace functions generated by the operators  $L_\gamma^t$  as different representations  $\pi_t$  of an abstract algebra  $\mathcal{A}_b$  which does not depend on the choice of a triangulation,  $L_\gamma^t \equiv \pi_t(L_\gamma)$  for  $L_\gamma \in \mathcal{A}_b$ .

The operators  $L_\gamma^t$  are positive self-adjoint with spectrum bounded from below by 2, as follows from the result of [Ka4]. Two operators  $L_{\gamma_1}^t$  and  $L_{\gamma_2}^t$  commute if the intersection of  $\gamma_1$  and  $\gamma_2$  is empty. It is therefore possible to diagonalise simultaneously the quantised trace functions associated to a maximal set of non-intersecting closed curves defining a pants decomposition [T05, TV13].

### 5.3 Representations associated to pants decompositions

Mutual commutativity of the quantized trace-functions  $L_{\gamma_r}^t$  ensures existence of operators  $R_{\sigma|t}$  which map the operators  $L_{\gamma_r}^t$ ,  $r = 1, \dots, h$  associated to the curves  $\mathcal{C} = (\gamma_1, \dots, \gamma_h)$  defining a pants decomposition to the operators of multiplication by  $2 \cosh(l_r/2)$ . The states in the image  $\mathcal{H}_\sigma$  of  $R_{\sigma|t}$  can be represented by functions  $\psi(l)$ ,  $l = (l_1, \dots, l_h)$  depending on variables  $l_r \in \mathbb{R}^+$  which parameterise the eigenvalues of  $L_{\gamma_r}^t$ . The operators  $R_{\sigma|t}$  define a new family of representations  $\pi_\sigma$  of  $\mathcal{A}_b$  via

$$\pi_\sigma(L_\gamma) := R_{\sigma|t} \cdot \pi_t(L_\gamma) \cdot (R_{\sigma|t})^{-1} . \quad (5.6)$$

The representations are naturally labelled by the data  $\sigma = (\mathcal{C}, \Gamma)$  we had encountered before. The unitary operators  $R_{\sigma|t} : \mathcal{H}_t \rightarrow \mathcal{H}_\sigma$  were constructed explicitly in [T05, TV13].

#### 5.3.1 Transitions between representation

The passage between the representations  $\pi_{\sigma_1}$  and  $\pi_{\sigma_2}$  associated to two different pants decompositions is then described by

$$U_{\sigma_2\sigma_1} := R_{\sigma_2|t} \cdot (R_{\sigma_1|t})^{-1} .$$

The unitary operators  $U_{\sigma_2\sigma_1}$  intertwine the representations  $\pi_{\sigma_1}$  and  $\pi_{\sigma_2}$ ,

$$\pi_{\sigma_2}(L_\gamma) \cdot U_{\sigma_2\sigma_1} = U_{\sigma_2\sigma_1} \cdot \pi_{\sigma_1}(L_\gamma) . \quad (5.7)$$

Explicit representations for the operators  $U_{\sigma_2\sigma_1}$  have been calculated in [NT, TV13] for pairs  $[\sigma_2, \sigma_1]$  related by the generators of the Moore-Seiberg groupoid. The B-move is represented as

$$(B\psi_s)(\beta) = B_{l_2 l_1}^{l_3} \psi_s(\beta) , \quad B_{l_2 l_1}^{l_3} = e^{\pi i (\Delta_{l_3} - \Delta_{l_2} - \Delta_{l_1})} , \quad (5.8)$$

where  $\Delta_l = (1 + b^2)/4b + (l/4\pi b)^2$ . The F-move is represented in terms of an integral transformation of the form

$$\psi_s(l_s) \equiv (F\psi_t)(l_s) = \int_{\mathbb{R}^+} dl_t F_{l_s l_t} \begin{bmatrix} l_3 & l_2 \\ l_4 & l_1 \end{bmatrix} \psi_t(l_t). \quad (5.9)$$

A similar formula exists for the S-move. The explicit expressions can be found in [TV13].

The operators  $U_{\sigma_2\sigma_1}$  define a unitary projective representation of the Moore-Seiberg groupoid,

$$U_{\sigma_3\sigma_2} \cdot U_{\sigma_2\sigma_1} = \zeta_{\sigma_3\sigma_2\sigma_1} U_{\sigma_3\sigma_1}, \quad (5.10)$$

where  $\zeta_{\sigma_3\sigma_2\sigma_1} \in \mathbb{C}$ ,  $|\zeta_{\sigma_3\sigma_2\sigma_1}| = 1$ . The explicit formulae for the relations of the Moore-Seiberg groupoid in the quantisation of  $\mathcal{M}_{\text{flat}}^0(C)$  are listed in [TV13].

Having a representation of the Moore-Seiberg groupoid automatically produces a representation of the mapping class group. An element of the mapping class group  $\mu$  represents a diffeomorphism of the surface  $C$ , and therefore maps any MS graph  $\sigma$  to another one denoted  $\mu.\sigma$ . Note that the Hilbert spaces  $\mathcal{H}_\sigma$  and  $\mathcal{H}_{\mu.\sigma}$  are canonically isomorphic. Indeed, the Hilbert spaces  $\mathcal{H}_\sigma$  depend only on the combinatorics of the graphs  $\sigma$ , but not on their embedding into  $C$ . We may therefore define an operator  $M_\sigma(\mu) : \mathcal{H}_\sigma \rightarrow \mathcal{H}_\sigma$  as

$$M_\sigma(\mu) := U_{\mu.\sigma,\sigma}. \quad (5.11)$$

It is automatic that the operators  $M(\mu)$  define a projective unitary representation of the mapping class group  $\text{MCG}(C)$  on  $\mathcal{H}_\sigma$ .

*The kernels of the operators  $U_{\sigma_2\sigma_1}$ ,  $T_{t_2t_1}$  and  $R_{\sigma|t}$  are related to the partition functions of  $d = 3$  gauge theories on duality walls, see [DGV],[V:10] and references therein. The relations are summarised in the following table:*

Riemann surface $C$	Gauge theory $\mathcal{G}_C$
Kernels representing operators $U_{\sigma_2\sigma_1}$	UV duality walls $T_2[M, \mathbf{p}, \mathbf{p}']$
Kernels representing operators $R_{\sigma t}$	RG domain walls $T_2[M, \mathbf{p}, \Pi]$
Kernels representing operators $T_{t_2t_1}$	IR duality walls $T_2[M, \Pi, \Pi']$

### 5.3.2 Representations

The representations  $\pi_\sigma(L_\gamma)$  were calculated explicitly for the generators of  $\mathcal{A}_b$  in [TV13].

As a prototypical example let us consider the case where  $\sigma$  corresponds to the pants decomposition of  $C_{0,4}$  depicted on the left of Figure 1. We may associate generators  $L_s$ ,  $L_t$  and  $L_u$

of  $\mathcal{A}_b$  to the simple closed curves  $\gamma_s$ ,  $\gamma_t$ , and  $\gamma_u$  introduced in Subsection B.6, respectively. The generators  $L_r$   $r = 1, \dots, 4$  are associated to the boundary components of  $C \simeq C_{0,4}$ . The representation of  $\mathcal{A}_b$  will be generated from the operators  $L_s$ ,  $L_t$  and  $L_u$  defined as follows:

$$L_s := 2 \cosh(l/2). \quad (5.12a)$$

$$L_t := \frac{1}{2(\cosh l_s - \cos 2\pi b^2)} \left( 2 \cos \pi b^2 (L_2 L_3 + L_1 L_4) + L_s (L_1 L_3 + L_2 L_4) \right) \quad (5.12b)$$

$$+ \sum_{\epsilon=\pm 1} \frac{1}{\sqrt{2 \sinh(l_s/2)}} e^{\epsilon k/2} \frac{\sqrt{c_{12}(L_s) c_{34}(L_s)}}{2 \sinh(l_s/2)} e^{\epsilon k/2} \frac{1}{\sqrt{2 \sinh(l_s/2)}}$$

where

$$l \psi_\sigma(l) = l_s \psi_\sigma(l), \quad k \psi_\sigma(l) = -4\pi i b^2 \partial_l \psi_\sigma(l),$$

and  $c_{ij}(L_s)$  is defined as

$$c_{ij}(L_s) = L_s^2 + L_i^2 + L_j^2 + L_s L_i L_j - 4. \quad (5.12c)$$

$L_u$  is given by a similar expression [TV13]. The operators  $l_s$  and  $k_s$  are quantum counterparts of the Fenchel-Nielsen coordinates, see Appendix B.8 for a definition.

*As indicated above, one may interpret the trace functions  $L_s$ ,  $L_t$ ,  $L_u$  as UV line operators, here for the  $N_f = 4$  theory associated to  $C_{0,4}$ .  $L_s$ ,  $L_t$  and  $L_u$  correspond to the Wilson loop, 't Hooft loop, and simplest dyonic loop, respectively. The formulae above are directly related to the expectation values of these line operators on  $\mathbb{R}^3 \times_b S^1$  calculated in [IOT].*

### 5.3.3 The algebra of trace functions

Using the explicit representations for the generators of  $\mathcal{A}_b$  obtained in [TV13] it becomes straightforward to calculate the relations that they satisfy. As a prototypical example, let us again consider the case  $C = C_{0,4}$ . There are two main relations:

Quadratic relation:

$$\mathcal{Q}(L_s, L_t, L_u) := e^{\pi i b^2} L_s L_t - e^{-\pi i b^2} L_t L_s \quad (5.13)$$

$$- (e^{2\pi i b^2} - e^{-2\pi i b^2}) L_u - (e^{\pi i b^2} - e^{-\pi i b^2}) (L_1 L_3 + L_2 L_4).$$

Cubic relation:

$$\mathcal{P}(L_s, L_t, L_u) = -e^{\pi i b^2} L_s L_t L_u \quad (5.14)$$

$$+ e^{2\pi i b^2} L_s^2 + e^{-2\pi i b^2} L_t^2 + e^{2\pi i b^2} L_u^2$$

$$+ e^{\pi i b^2} L_s (L_3 L_4 + L_1 L_2) + e^{-\pi i b^2} L_t (L_2 L_3 + L_1 L_4) + e^{\pi i b^2} L_u (L_1 L_3 + L_2 L_4)$$

$$+ L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_1 L_2 L_3 L_4 - (2 \cos \pi b^2)^2.$$

The generators  $L_k$ ,  $k = 1, \dots, 4$  are central elements in  $\mathcal{A}_b(C_{0,4})$ , associated to the boundary components. The quadratic relations represent the deformation of the Poisson bracket (B.15), while the cubic relation is a deformation of the relation (B.8).

## 6. Relation to Liouville theory

Having worked out the quantization of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$ , we have determined the monodromy data we need to define the Riemann-Hilbert type problem discussed in Section 3.2. In order to derive the AGT-correspondence along these lines it remains to observe that the Liouville conformal blocks provide solutions to this Riemann-Hilbert problem.

Our goal in this section is to explain why Liouville conformal blocks are the wave-functions solving the Riemann-Hilbert problem (3.11). To this aim we are going to explain that

Liouville theory is just another way to represent the quantum theory of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  defined in Section 5.

The identification between conformal blocks and wave-functions in the quantum theory of moduli spaces of flat connections will follow naturally.

### 6.1 Complex-analytic Darboux coordinates for $\mathcal{M}_{\text{flat}}^0(C)$

Our explanations will be based on the fact that  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  is isomorphic to the Teichmüller space  $\mathcal{T}(C)$  (see Appendix B.2). This implies that there exists an alternative quantisation scheme using *holomorphic* coordinates for  $\mathcal{T}(C)$ . We are going to explain that the quantum theory in the resulting quantisation scheme is naturally related to conformal field theory.

For simplicity, we will here restrict attention to  $C = C_{0,4} = \mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$ . We do not lose generality when we assume that  $z_1 = 0$ ,  $z_3 = 1$ ,  $z_4 = \infty$ . The value of  $q := z_2$  defines a complex-analytic coordinate for the moduli space  $\mathcal{M}(C)$  of complex structures on  $C$ . The Fuchsian group corresponding to the complex structure parameterized by a value of  $q$  defines a flat  $\text{PSL}(2, \mathbb{R})$ -connection. We may therefore regard  $q$  as a local coordinate for  $\mathcal{M}_{\text{flat}}^0(C)$  which is related to the Fenchel-Nielsen coordinates  $(k, l)$  in a very complicated way. The relation becomes reasonably simple only in the limit  $|q| \rightarrow 0 \Leftrightarrow l \rightarrow 0$ , where one has

$$\frac{l}{2\pi} \simeq \frac{\pi}{\log(1/|q|)}, \quad 2\pi k \simeq \arg(q), \quad (6.1)$$

where the notation  $\simeq$  indicates equality to leading order in this limit.

The complicated nature of the dependence of  $q$  on the Darboux coordinates  $(k, l)$  is reflected in the fact that the Poisson structure on  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  is represented in terms of  $q$  in a much more

complicated way. A useful way to describe the Poisson structure using the coordinate  $q$  is to find a function  $h = h(q, \bar{q})$  that is canonically conjugate to  $q$  in the sense that

$$\{q, h(q, \bar{q})\} = -i. \quad (6.2)$$

Such a function can be found from the metric  $ds^2 = e^{2\varphi} dy d\bar{y}$  of constant negative curvature associated to  $q$  by writing the function  $t(y) = -(\partial_y \varphi)^2 + \partial_y^2 \varphi$  in the form

$$t(y) = \frac{\delta_3}{(y-1)^2} + \frac{\delta_1}{y^2} + \frac{\delta_2}{(y-q)^2} + \frac{v}{y(y-1)} + \frac{q(q-1)}{y(y-1)} \frac{h}{y-q}. \quad (6.3)$$

The residue  $h = h(q, \bar{q})$  in (6.3) is indeed the sought-for conjugate variable to  $q$ , as follows from the beautiful results [TZ87a, CMS, TZ03] that the classical Liouville action  $S_{\text{cl}}[\varphi]$  is the Kähler potential for the symplectic form on  $\mathcal{T}(C) \simeq \mathcal{M}_{\text{flat}}^0(C)$  corresponding to the Poisson-structure we consider, and that

$$h(q, \bar{q}) = -\frac{\partial}{\partial q} S_{\text{cl}}[\varphi]. \quad (6.4)$$

The function  $h(q, \bar{q})$  is called the accessory parameter. Having real monodromy (subgroup of  $\text{PSL}(2, \mathbb{R})$ ) of the differential operator  $\partial_y^2 + t$  clearly requires fine-tuning of the residue  $h$  in (6.3) in a way that depends on the complex structure  $q$ .

## 6.2 Quantization of complex-analytic Darboux coordinates for $\mathcal{M}_{\text{flat}}^0(C)$

One may then consider an alternative representation for the quantum theory of  $\mathcal{M}_{\text{flat}}^0(C)$  which is such that the operator representing the complex-analytic coordinate  $q$  is realized as a multiplication operator  $q$ ,

$$q \psi(q) = q \psi(q). \quad (6.5)$$

The quantization of the observable  $h$  should then give an operator  $h$  that satisfies

$$[h, q] = b^2, \quad (6.6)$$

and can therefore be represented as

$$h \psi(q) = b^2 \frac{\partial}{\partial q} \psi(q). \quad (6.7)$$

In order for such a representation to be equivalent to the representation we had previously defined using the Darboux coordinates  $(k, l)$  we should consider wave-functions  $\phi(q)$  that are holomorphic in  $q$ . Such a representation can be seen as an analog of the coherent state representation of quantum mechanics.

It will be useful for us to think of the wave-functions  $\psi(q)$  in such a representation as overlaps  $\langle q | \psi \rangle$  of the abstract state  $|\psi\rangle$  with an eigenstate  $\langle q |$  of the operator  $q$ .

### 6.3 Geometric definition of the conformal blocks

In order to see how the quantisation of  $\mathcal{T}(C)$  is related to conformal field theory, let us present a more geometric approach to the definition of the conformal blocks going back to [BPZ].

Let  $C$  be the Riemann surface  $C = \mathbb{P}^1 \setminus \{z_1, \dots, z_n\}$  of genus 0 with  $n$  marked points  $z_1, \dots, z_n$ . At each of the marked points  $z_r$ ,  $r = 1, \dots, n$ , let us choose the local coordinates  $w_r = y - z_r$ . We associate highest weight representations  $\mathcal{V}_r$ , of  $\text{Vir}_c$  to  $P_r$ ,  $r = 1, \dots, n$ . The representations  $\mathcal{V}_r$  are generated from highest weight vectors  $e_r$  with weights  $\Delta_r$ .

The conformal blocks are then defined to be the linear functionals  $\mathcal{F} : \mathcal{V}_{[n]} \equiv \otimes_{r=1}^n \mathcal{V}_r \rightarrow \mathbb{C}$  that satisfy the invariance property

$$\mathcal{F}(T[\chi] \cdot v) = 0 \quad \forall v \in \mathcal{V}_{[n]}, \quad \forall \chi \in \mathfrak{V}_{\text{out}}, \quad (6.8)$$

where  $\mathfrak{V}_{\text{out}}$  is the Lie algebra of meromorphic differential operators on  $C$  which may have poles only at  $z_1, \dots, z_n$ . The action of  $T[\chi]$  on  $\otimes_{r=1}^n \mathcal{R}_r \rightarrow \mathbb{C}$  is defined as

$$T[\chi] = \sum_{r=1}^n \text{id} \otimes \dots \otimes L[\chi^{(r)}] \otimes \dots \otimes \text{id}, \quad L[\chi^{(r)}] := \sum_{k \in \mathbb{Z}} L_k \chi_k^{(r)} \in \text{Vir}_c, \quad (6.9)$$

where  $\chi_k^{(r)}$  are the coefficients of the Laurent expansions of  $\chi$  at the points  $P_1, \dots, P_n$ ,

$$\chi(z_r) = \sum_{k \in \mathbb{Z}} \chi_k^{(r)} w_r^{k+1} \partial_{w_r} \in \mathbb{C}((w_r)) \partial_{w_r}, \quad (6.10)$$

with  $\mathbb{C}((t))$  being the space of Laurent series in the variable  $t$ .

The vector space of conformal blocks associated to the Riemann surface  $C$  with representations  $\mathcal{V}_r$  associated to the marked points  $P_r$ ,  $r = 1, \dots, n$  will be denoted as  $\text{CB}(\mathcal{V}_{[n]}, C)$ . It is the space of solutions to the defining invariance conditions (6.8).

The space  $\text{CB}(\mathcal{V}_{[n]}, C)$  is infinite-dimensional in general. Considering the case  $n = 4$ , for example, one may see this more explicitly by noting that the defining invariance property allows us to express the values  $\mathcal{F}(v_4 \otimes v_3 \otimes v_2 \otimes v_1)$  in terms of the complex numbers

$$\mathcal{Z}^{(k)}(\mathcal{F}, C) := \mathcal{F}(e_4 \otimes e_3 \otimes L_{-1}^k e_2 \otimes e_1), \quad k \in \mathbb{Z}^{>0}, \quad (6.11)$$

where  $e_i$  are the highest weight vectors of  $\mathcal{V}_i$ ,  $i = 1, 2, 3, 4$ . We note that  $\mathcal{F}$  is completely defined by the values  $\mathcal{Z}^{(k)}(\mathcal{F}, C)$ . The space of conformal blocks  $\text{CB}(\mathcal{V}_{[4]}, C)$  is therefore isomorphic as a vector space to the space of *formal* power series in one variable.

This definition of conformal blocks is closely related, but not quite identical to the one introduced previously in Section 2.5. To indicate the relation let us note that matrix elements like

$$\mathcal{F}_\beta(v_4 \otimes v_3 \otimes v_2 \otimes v_1) := \langle e_{\alpha_4}, V_{\alpha_4, \beta}^{\alpha_3}(z_3) V_{\beta, \alpha_1}^{\alpha_2}(z_2) e_{\alpha_1} \rangle_{\alpha_4}, \quad (6.12)$$

will represent particular examples for conformal blocks as defined in this section. Validity of the defining invariance property (6.8) follows from the covariance properties of the chiral vertex operators.

#### 6.4 Deformations of the complex structure of $C$

A key point that needs to be discussed about the spaces of conformal blocks is the dependence on the complex structure of  $C$ , here specified by the positions  $z_1, \dots, z_n$  of the marked points. There is a natural way to represent infinitesimal variations of the complex structure of  $C$  on the spaces of conformal blocks. By combining the definition of conformal blocks with the so-called “Virasoro uniformization” of the moduli space  $\mathcal{M}_{0,n}$  of complex structures on  $C = C_{0,n}$  one may construct a natural representation of infinitesimal motions on  $\mathcal{M}_{0,n}$  on the space of conformal blocks.

The “Virasoro uniformization” of the moduli space  $\mathcal{M}_{0,n}$  may be formulated as the statement that the tangent space  $T\mathcal{M}_{0,n}$  to  $\mathcal{M}_{0,n}$  at  $C$  can be identified with the double quotient

$$T\mathcal{M}_{0,n} = \Gamma(C \setminus \{z_1, \dots, z_n\}, \Theta_C) \Big/ \bigoplus_{k=1}^n \mathbb{C}((w_k))\partial_k \Big/ \bigoplus_{k=1}^n w_k \mathbb{C}[[w_k]]\partial_k, \quad (6.13)$$

where  $\mathbb{C}[[w_k]]$  are the spaces of Taylor series in the local coordinates  $w_k$  for  $k = 1, \dots, n$ , respectively, and  $\Gamma(C \setminus \{z_1, \dots, z_n\}, \Theta_C)$  is the space of vector fields that are holomorphic on  $C \setminus \{z_1, \dots, z_n\}$ , embedded into  $\bigoplus_{k=1}^n \mathbb{C}((w_k))\partial_k$  via (6.10).

Given a tangent vector  $\vartheta \in T\mathcal{M}_{0,n}$ , it follows from the Virasoro uniformization (6.13) that we may find elements  $\eta_\vartheta$  of  $\bigoplus_{k=1}^n \mathbb{C}((t_k))\partial_k$ , which represent  $\vartheta$  via (6.13). Let us then consider  $\mathcal{F}(T[\eta_\vartheta]v)$  with  $T[\eta]$  being defined in (6.9) in the case that  $v$  is the product of highest weight vectors,  $v = e_n \otimes \dots \otimes e_1$ . (6.13) allows us to define the derivative  $\delta_\vartheta \mathcal{F}(v)$  of  $\mathcal{F}(v)$  in the direction of  $\vartheta \in T\mathcal{M}_{0,n}$  as

$$\delta_\vartheta \mathcal{F}(v) := \mathcal{F}(T[\eta_\vartheta]v), \quad (6.14)$$

Dropping the condition that  $v$  is a product of highest weight vectors, one may use (6.14) to define  $\delta_\vartheta \mathcal{F}$  in general. And indeed, it is well-known that (6.14) leads to the definition of a canonical flat connection on the space  $\text{CB}(\mathcal{V}_{[n]}, C)$  of conformal blocks [BF].

#### 6.5 Conformal blocks versus function on $\mathcal{T}_{0,n}$

In the case  $n = 4$  it is easy to see that (6.14) can be reduced simply to

$$\partial_z \mathcal{F}(v_4 \otimes v_3 \otimes v_2 \otimes v_1) = \mathcal{F}(v_4 \otimes v_3 \otimes L_{-1}v_2 \otimes v_1). \quad (6.15)$$



Let us introduce the notation

$$\mathcal{Z}^{\text{Liou}}(\mathcal{F}, C) = \mathcal{F}(e_1 \otimes \dots \otimes e_n), \quad (6.16)$$

for the value of  $\mathcal{F}$  on the product of highest weight vectors. Equation (6.15) allows us to identify the values  $\mathcal{Z}^{(k)}(\mathcal{F}, C)$  defined in (6.11) as the  $k$ -th derivatives of the partition functions  $\mathcal{Z}^{\text{Liou}}(\mathcal{F}, C)$ . We had seen above that the collection of the numbers  $\mathcal{Z}^{(k)}(\mathcal{F}, C)$  characterizes the conformal blocks  $\mathcal{F} \in \text{CB}(\mathcal{V}_{[4]}, C)$  completely.

One may define the parallel transport of conformal blocks over  $\mathcal{M}_{0,4}$  via

$$\mathcal{Z}^{\text{Liou}}(\mathcal{F}, C_w) = \sum_{k=0}^{\infty} \frac{1}{k!} (w - z)^k \mathcal{Z}^{(k)}(\mathcal{F}, C_z). \quad (6.17)$$

We see that there is a one-to-one correspondence between functions  $\mathcal{Y}(w)$  defined on some open, simply connected neighborhood  $\mathcal{U}_z$  of a point  $z$  in  $\mathcal{T}_{0,4}$  and conformal blocks  $\mathcal{F}$  for which the series (6.17) converges for all  $w \in \mathcal{U}_z$ : The Taylor expansion coefficients  $\mathcal{Y}_k$  of  $\mathcal{Y}(w)$  can be used to define a conformal block  $\mathcal{F}_{\mathcal{Y}} \in \text{CB}(\mathcal{V}_{[4]}, C)$  such that  $\mathcal{Z}^{(k)}(\mathcal{F}_{\mathcal{Y}}, C) = \mathcal{Y}_k$ . Conversely, for “well-behaved” conformal blocks  $\mathcal{F} \in \text{CB}(\mathcal{V}_{[4]}, C_z)$  one may use (6.17) to define a family of conformal blocks in a neighborhood  $\mathcal{U}_z(\mathcal{F})$ .

We are ultimately not interested in the most crazy conformal blocks, but rather in those whose partition functions can be analytically continued over all of  $\mathcal{T}(C)$ , and which have reasonably mild singular behaviour at the boundaries of  $\mathcal{T}(C)$ . Such a subspace will be denoted  $\text{CB}^{\text{reg}}(\mathcal{V}_{[4]}, C)$ . It was proposed in [TV13] that the conformal blocks defined previously with the help of chiral vertex operators generate a basis for  $\text{CB}^{\text{reg}}(\mathcal{V}_{[4]}, C)$  in a suitable sense. This proposal is based on the highly nontrivial results of [T01, T03a] that the partition functions  $\mathcal{Z}^{\text{Liou}}$  can be analytically continued over all of  $\mathcal{T}(C)$ , and that the bases associated to different pants decompositions are linearly related.

## 6.6 Verlinde loop operators

The construction of conformal blocks using chiral vertex operators, or more generally by gluing conformal blocks associated to three-punctured spheres gives another way to define a natural family of operators acting on spaces of conformal blocks. The resulting operators will be identified with quantized trace functions. We will describe the construction in the case of genus 0 in terms of chiral vertex operators.

Let us consider chiral vertex operators  $V_{\beta_2\beta_1}^{\alpha}(z)$  in the special case where  $\alpha = -b/2$ , assuming that  $Q$  is represented as  $Q = b + b^{-1}$ . If furthermore  $\beta_2$  and  $\beta_1$  are related as  $\beta_2 = \beta_1 \mp b/2$ , the vertex operators  $\psi_s(y) \equiv \psi_{\beta_1, s}(y) := V_{\beta_1 - sb/2, \beta_1}^{-b/2}(y)$ ,  $s \in \{1, -1\}$ , are well-known to satisfy a

differential equation of the form

$$\partial_y^2 \psi_{\beta_1, s}(y) + b^2 : T(y) \psi_{\beta_1, s}(y) := 0, \quad (6.18)$$

with normal ordering defined in (2.18). The chiral vertex operators  $\psi_{\beta_1, s}(y)$  are called degenerate fields. It follows from (6.18) that matrix elements such as

$$\begin{aligned} \mathcal{F}_{ss'}(\alpha; \beta | z | y_0 | y) &:= \langle \alpha_4 | \psi_s(y_0) \psi_{s'}(y) | \Theta_{s+s'} \rangle, \\ | \Theta_{s+s'} \rangle &:= V_{\alpha_4 + (s+s')\frac{b}{2}, \beta}^{\alpha_3}(z_3) V_{\beta \alpha_1}^{\alpha_2}(z_2) V_{\alpha_1, 0}^{\alpha_1}(z_1) | 0 \rangle, \end{aligned} \quad (6.19)$$

will satisfy the partial differential equation  $\mathcal{D}_{\text{BPZ}} \mathcal{F} = 0$ , with

$$\mathcal{D}_{\text{BPZ}} := \frac{1}{b^2} \frac{\partial^2}{\partial y^2} + \frac{\Delta_{-\frac{b}{2}}}{(y - y_0)^2} + \frac{1}{y - y_0} \frac{\partial}{\partial y_0} + \sum_{k=1}^3 \left( \frac{\Delta_{\alpha_k}}{(y - z_k)^2} + \frac{1}{y - z_k} \frac{\partial}{\partial z_k} \right). \quad (6.20)$$

As explained previously, we may regard the matrix elements (6.19) as the partition functions of conformal blocks in  $\text{CB}(\mathcal{V}'_{[6]}, C_{0,6})$ , where now  $\mathcal{V}'_{[6]} = \mathcal{V}_{[4]} \otimes \mathcal{V}_{-b/2}^{\otimes 2}$ .

Using these ingredients it is straightforward to show that the analytic continuation of the matrix elements  $\mathcal{F}_{ss'}(\alpha; \beta | z | y_0 | y)$ ,  $s, s' \in \{1, -1\}$  with respect to  $y$  along closed paths  $\gamma$  on  $C_{0,5}$  can be expressed as a linear combination of the matrix elements  $\mathcal{F}_{ss''}(\alpha; \beta' | z | y_0 | y)$  having parameters  $\beta'$  that differ from  $\beta$  by integer multiples of the parameter  $b$ ,

$$\mathcal{F}_{ss'}(\alpha; \beta | z | y_0 | y) = \sum_{s''=\pm} \mathbf{M}_\gamma(\beta, \mathbf{T}_\beta)_{ss''}^{s''} \cdot \mathcal{F}_{ss''}(\alpha; \beta' | z | y_0 | y) \quad (6.21)$$

where  $\mathbf{T}$  is the operator which shifts the argument  $\beta$  of  $\mathcal{F}_{ss'}$  by the amount  $b$ . The matrices  $\mathbf{M}_\gamma$  define representations of the fundamental group  $\pi_1(C_{0,5})$  on the space  $\text{CB}(\mathcal{V}'_{[6]}, C_{0,6})$ .

The definition of the Verlinde loop operators is based on the simple fact that  $\text{CB}(\mathcal{V}_{[4]} \otimes \mathcal{V}_0, C_{0,5})$  is canonically isomorphic to  $\text{CB}(\mathcal{V}_{[4]}, C_{0,4})$  if  $\mathcal{V}_0$  is the vacuum representation. One may furthermore note that there exists a linear combination  $\sum_s K_s \psi_s(y_0) \psi_{-s}(y)$  which is a descendant of the chiral vertex operator  $V_{\beta, \beta}^0[\psi_+(y_0 - y) e_{-\frac{b}{2}}](y)$  associated to the vacuum representation. These observations allow us to define both an embedding  $\iota$  and a projection  $\wp$ ,

$$\begin{aligned} \iota : \text{CB}^{\text{reg}}(\mathcal{V}_{[4]}, C_{0,4}) &\hookrightarrow \text{CB}^{\text{reg}}(\mathcal{V}'_{[6]}, C_{0,6}), \\ \wp : \text{CB}^{\text{reg}}(\mathcal{V}'_{[6]}, C_{0,6}) &\rightarrow \text{CB}^{\text{reg}}(\mathcal{V}_{[4]}, C_{0,4}), \end{aligned} \quad (6.22)$$

in a natural way. The Verlinde loop operators can then be defined as the composition

$$\mathbf{V}_\gamma := \wp \circ \mathbf{M}_\gamma \circ \iota. \quad (6.23)$$

Concretely this boils down to taking a certain linear combination of the matrix elements  $\mathbf{M}_\gamma(\beta, \mathbf{T}_\beta)_{ss''}^{s''}$  representing the monodromy along  $\gamma$  on  $\text{CB}^{\text{reg}}(\mathcal{V}'_{[6]}, C_{0,6})$ . The explicit calculations of the operator  $\mathbf{V}_\gamma$  in [AGGTV, DGOT] shows that the Verlinde loop operators define a

representation of  $\text{Fun}_q(\mathcal{M}_{\text{flat}}(C))$  on the space of conformal blocks  $\text{CB}^{\text{reg}}(\mathcal{V}_{[4]}, C_{0,n})$  which is equivalent to the one defined in Section 5.3.2 if we identify variables

$$\beta = \frac{Q}{2} + i\frac{l}{4\pi b}, \quad \alpha_k = \frac{Q}{2} + i\frac{l_k}{4\pi b}, \quad k = 1, \dots, 4. \quad (6.24)$$

Let us summarize the observations made in this section in the following table:

Quantisation of ...	... is realised in CFT via
Darboux coordinates $(q, h)$	conformal Ward identities
Fenchel-Nielsen coordinates $(k, l)$	Verlinde loop operators

*The degenerate fields have a beautiful interpretation in the gauge theories  $\mathcal{G}_C$  in terms of a family of observables called surface operators [AGGTV], explained also in the article [V:7] in this collection.*

## 6.7 Liouville conformal blocks as solutions to the Riemann-Hilbert problem

We claim that the solution to the Riemann-Hilbert type problem defined in Section 3.2 is given by the Liouville conformal blocks in the following sense

$$\mathcal{Z}_{\sigma}^{\text{inst}}(a, m; \tau; \epsilon_1, \epsilon_2) = \mathcal{Z}_{\sigma}^{\text{spur}}(\alpha; \tau; b) \mathcal{Z}_{\sigma}^{\text{Liou}}(\beta, \alpha; q; b), \quad (6.25)$$

The solution of the Riemann-Hilbert problem defined in Section 3.2 is unique up to multiplication with meromorphic functions which may have poles only at the boundary of  $\mathcal{M}(C_{0,4})$ . The resulting freedom can be absorbed into  $\mathcal{Z}_{\sigma}^{\text{spur}}(\alpha, \tau; b)$ .

In order to verify (6.25) we need to show that the representation of the mapping class group on spaces of Liouville conformal blocks is the same as the one coming from the quantum theory of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  as described in Section 5. This boils down to the comparison of the respective realizations of  $B$  and  $F$ -moves. The coincidence of  $B$ -moves is trivial to verify. The realization of the  $F$ -move on Liouville conformal blocks was calculated in [T01], where a relation of the form

$$\mathcal{Z}_s^{\text{Liou}}(\beta_1, q) = \int_{\mathbb{S}} d\beta_2 F_{\beta_1\beta_2} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} \mathcal{Z}_t^{\text{Liou}}(\beta_2, q), \quad \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}^+, \quad (6.26)$$

was found. For the normalization defined in (2.14) we find the *same* kernel  $F_{\beta_1\beta_2} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}$  in the relation (6.26), as was found within the quantum theory of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  described in Section 5.

This is good enough to conclude that (6.25) must hold. To round off the picture, let us exhibit the meaning of the partition functions  $\mathcal{Z}_{\sigma}^{\text{Liou}}(\beta, \alpha; q; b)$  within the quantisation of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$ .

In this section we have described two different representations for the quantisation of one and the same Poisson-manifold, obtained by quantisation of the coordinates  $(q, h)$  and  $(l, k)$ , respectively. One may expect that these two representations should be unitarily equivalent. The eigenstates  $|l\rangle$  of the operator  $l$  are complete in  $\mathcal{H}_{\sigma_s}$ . It should therefore be possible to relate the wave-function  $\psi(q) \equiv \langle q|\psi\rangle$  representing a state  $|\psi\rangle$  in the holomorphic representation to the wave-function  $\Psi(l) \equiv \langle l|\psi\rangle$  representing the same state in the length representation as

$$\psi(q) \equiv \langle q|\psi\rangle = \int dl \langle q|l\rangle \langle l|\psi\rangle. \quad (6.27)$$

The kernel  $\langle q|l\rangle$  is the complex conjugate of the wave-function  $\langle l|q\rangle$  of the “coherent” state  $|q\rangle$  in the length representation.

One may use essentially the same arguments as presented in Section 3.2 to conclude that  $\langle q|l\rangle$  must solve the same Riemann-Hilbert problem as discussed above. Combined with a discussion of the asymptotics at the boundary of  $\mathcal{T}(C)$  we may thereby conclude [T03b, TV13] that

$$\langle q|l\rangle = \mathcal{Z}_{\sigma}^{\text{Liou}}(\beta, \alpha; q; b). \quad (6.28)$$

We have thereby identified more precisely which wave-functions the conformal blocks are: They describe the change of representation between the two natural representations for the quantum theory of  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$  discussed in this section.

## 6.8 The Nekrasov-Shatashvili limit

The results reviewed in this article are related to the work [NRS] in an interesting way. In order to explain the relations to [NRS] let us consider the limit  $\epsilon_2 \rightarrow 0$  corresponding to the classical limit for the quantum theory discussed in the previous sections. This will also provide further insight into the meaning of  $\langle q|l\rangle = \mathcal{Z}^{\text{Liou}}$ . It can be shown [T10] that the conformal blocks behave as

$$\log \mathcal{Z}_{\sigma}^{\text{Liou}}(\beta, \alpha; q; b) \sim -\frac{1}{\epsilon_2} \mathcal{Y}(l, m; q; \epsilon_1), \quad (6.29)$$

assuming that the variables are related by (2.16).  $\mathcal{Y}(l, m; q; \epsilon_1)$  is defined as follows: For given values of  $l$  and  $q$  let us consider differential operators of the form  $\epsilon_1^2(\partial_y^2 + t(y))$ , with  $t(y)$  of the form (6.3). It can be argued that there is a unique choice  $h = h(l, q)$  for the residue  $h$  such that  $\epsilon_1^2(\partial_y^2 + t(y))$  has monodromy with trace equal to  $2 \cosh(l/2)$ .  $\mathcal{Y}(l, m; q; \epsilon_1)$  is defined up to a constant by the condition that

$$\frac{\partial}{\partial q} \mathcal{Y}(l, m; q; \epsilon_1) = -h(l, q). \quad (6.30)$$

The constant can be fixed by demanding that the constant term  $\mathcal{Y}_0(l, m; \epsilon_1)$  in

$$\mathcal{Y}(l, m; q; \epsilon_1) \underset{q \rightarrow 0}{\sim} -(\delta - \delta_1 - \delta_2) \log q + \mathcal{Y}_0(l, m; \epsilon_1) + \mathcal{O}(q), \quad (6.31)$$

is one half of the sum of the Liouville actions on the three-punctured spheres into which  $C_{0,4}$  splits when  $q \rightarrow 0$ . We furthermore have

$$k(l, q) = 4\pi i \frac{\partial}{\partial l} \mathcal{Y}(l, m; q; \epsilon_1). \quad (6.32)$$

To verify (6.32) note, on the one hand, that the Verlinde loop operators reduce to the trace functions in the limit  $\epsilon_2 \rightarrow 0$ . Recall that the trace functions may be parameterised by the (complexified) Fenchel-Nielsen coordinates  $(l, k)$ . The resulting expression may be compared to one following from (5.12) and (6.29) in this limit, giving (6.32).

The pairs of coordinates  $(l, k(l, q))$  describe a Lagrangian sub-manifold denoted  $\text{Op}_{\text{sl}_2}(C)$  within  $\mathcal{M}_{\text{flat}}(C)$  sometimes called the "brane of opers". It follows from (6.32), (6.30) that  $\mathcal{Y}(l, m; q; \epsilon_1)$  is the generating function of this sub-manifold. We thereby arrive at the description for the  $\epsilon_2$ -limit of the instanton partition functions that was proposed in [NRS]. One may therefore view the results of [TV13] reviewed in this article as the generalisation of the results from [NRS] to nonzero  $\epsilon_2$ .

## 6.9 Quantization of Seiberg-Witten theory

It will furthermore be instructive to consider the limit where both  $\epsilon_1, \epsilon_2 \rightarrow 0$ , in which  $E_{\epsilon_1, \epsilon_2}^4$  turns into  $\mathbb{R}^4$ , and we can make contact with Seiberg Witten theory.

To begin with, let us note that  $\epsilon_1 \partial_y^2 + t(y)$  turns into the quadratic differential  $-\vartheta(y)$  when  $\epsilon_1 \rightarrow 0$ . Using  $\vartheta(y)$  we define the Seiberg-Witten curve  $\Sigma$  as usual by

$$\Sigma = \{ (v, u) \mid v^2 = \vartheta(u) \}. \quad (6.33)$$

It follows by WKB analysis of the differential equation  $(\epsilon_1 \partial_y^2 + t(y))\chi = 0$  that the coordinates  $l_e$  have asymptotics that can be expressed in terms of the Seiberg-Witten differential  $\Lambda$  on  $\Sigma$  defined such that  $\Lambda^2 = \vartheta(u)(du)^2$ . We find

$$l \sim \frac{4\pi}{\epsilon_1} a, \quad k \sim \frac{4\pi}{\epsilon_1} a^D, \quad (6.34)$$

where  $a$  and  $a^D$  are periods of the Seiberg-Witten differential  $\Lambda$  defined as

$$a := \int_{\alpha} \Lambda, \quad a^D := \int_{\beta} \Lambda, \quad (6.35)$$

with  $\alpha$  and  $\beta$  being lifts of  $\gamma_s$  and  $\gamma_t$  to cycles on  $\Sigma$  that project to zero, respectively.

The prepotential  $\mathcal{F}$  is obtained in the limit  $\epsilon_1, \epsilon_2 \rightarrow 0$  as follows:

$$\begin{aligned} \mathcal{F}(a, m, q) &:= \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_1 \epsilon_2 \mathcal{Z}_{\sigma}^{\text{inst}}(a, m, q; \epsilon_1, \epsilon_2) \\ &= \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_1 \mathcal{Y}(a, m; q; \epsilon_1). \end{aligned} \quad (6.36)$$

$\mathcal{F}(a, m; q)$  satisfies the relations

$$a^D = \frac{\partial}{\partial a} \mathcal{F}(a, q), \quad h = -\frac{\partial}{\partial q} \mathcal{F}(a, q). \quad (6.37)$$

A proof of the relations (6.37) that is valid for all  $A_1$ -theories of class  $\mathcal{S}$  was given in [GT, Section 7.3.2]. The relations (6.37) are equivalent to the statement that both the coordinates  $(a, a^D)$  describing the special geometry underlying Seiberg-Witten theory, and the coordinates  $(q, h)$  introduced above can be seen as systems of Darboux coordinates for the same space  $T^*\mathcal{T}(C)$ . The prepotential  $\mathcal{F}(a, m, q)$  is nothing but the generating function of the change of variables between  $(a, a^D)$  and  $(q, h)$ .

These observations show that the relations between the quantum theory on  $\mathcal{M}_{\text{flat}}^0(C)$  and the gauge theories  $\mathcal{G}_C$  discussed in this article can be seen as the quantization of the special geometry used in Seiberg-Witten theory. The dual zero modes  $a$  and  $a^D$  turn into the Darboux coordinates  $l$  and  $k$  upon partial compactification to  $S^1 \times \mathbb{R}^3$  or  $E_{\epsilon_1}^2 \times \mathbb{R}^2$ . Further compactification to a four-ellipsoid leads to the quantization of these zero modes.

*It is intriguing to observe that very similar ideas have been discussed in the context of topological string theory, where it has been proposed that the partition function of the topological string has an interpretation as a wave-function arising from the quantization of special geometry. The geometric engineering of gauge theories within string theory leads to relations between topological string and instanton partition functions, see the articles [V:12, V:13] in this volume for a review. One may hope that the relations with the quantization of moduli spaces of vacua discussed in this article may help us to get a more unified picture.*

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## A. Riemann surfaces: Basic definitions and results

This appendix introduces basic definitions and results concerning Riemann surfaces  $C$  that will be used throughout the paper. A Riemann surface  $C$  is a two-dimensional topological surface  $S$  together with a choice of complex structure on  $S$ . We will denote by  $\mathcal{M}(S)$  the moduli space of complex structures on a two-dimensional surface  $S$ , and by  $\mathcal{T}(C)$  the Teichmüller space of deformations of complex structures on the Riemann surface  $C$ .

### A.1 Complex analytic gluing construction

A convenient family of particular coordinates on the Teichmüller spaces  $\mathcal{T}(C)$  is defined by means of the complex-analytic gluing construction of Riemann surfaces  $C$  from three punctured

spheres [Ma, HV]. Let us briefly review this construction.

Let  $C$  be a (possibly disconnected) Riemann surface. Fix a complex number  $q$  with  $|q| < 1$ , and pick two points  $Q_1$  and  $Q_2$  on  $C$  together with coordinates  $z_i(P)$  in a neighborhood of  $Q_i$ ,  $i = 1, 2$ , such that  $z_i(Q_i) = 0$ , and such that the discs  $D_i$ ,

$$D_i := \{ P_i \in C ; |z_i(P_i)| < |q|^{-\frac{1}{2}} \}, \quad i = 1, 2,$$

do not intersect. One may define the annuli  $A_i$ ,

$$A_i := \{ P_i \in C ; |q|^{\frac{1}{2}} < |z_i(P_i)| < |q|^{-\frac{1}{2}} \}, \quad i = 1, 2.$$

To glue  $A_1$  to  $A_2$  let us identify two points  $P_1$  and  $P_2$  on  $A_1$  and  $A_2$ , respectively, iff the coordinates of these two points satisfy the equation

$$z_1(P_1)z_2(P_2) = q. \quad (\text{A.1})$$

If  $C$  is connected one creates an additional handle, and if  $C = C_1 \sqcup C_2$  has two connected components one gets a single connected component after performing the gluing operation. In the limiting case where  $q = 0$  one gets a nodal surface which represents a component of the boundary  $\partial\mathcal{M}(S)$  defined by the Deligne-Mumford compactification  $\overline{\mathcal{M}}(S)$ .

By iterating the gluing operation one may build any Riemann surface  $C$  of genus  $g$  with  $n$  punctures from three-punctured spheres  $C_{0,3}$ . Embedded into  $C$  we naturally get a collection of annuli  $A_1, \dots, A_h$ , where

$$h := 3g - 3 + n. \quad (\text{A.2})$$

The construction above can be used to define a  $3g-3+n$ -parametric family of Riemann surfaces, parameterised by a collection  $q = (q_1, \dots, q_h)$  of complex parameters. These parameters can be taken as coordinates for a neighbourhood of a component in the boundary  $\partial\overline{\mathcal{M}}(S)$  which are complex-analytic with respect to its natural complex structure [Ma].

Conversely, assume given a Riemann surface  $C$  and a cut system, a collection  $\mathcal{C} = \{\gamma_1, \dots, \gamma_h\}$  of homotopy classes of non-intersecting simple closed curves on  $C$ . Cutting along all the curves in  $\mathcal{C}$  produces a pants decomposition,  $C \setminus \mathcal{C} \simeq \bigsqcup_v C_{0,3}^v$ , where the  $C_{0,3}^v$  are three-holed spheres.

Having glued  $C$  from three-punctured spheres defines a distinguished cut system, defined by a collection of simple closed curves  $\mathcal{C} = \{\gamma_1, \dots, \gamma_h\}$  such that  $\gamma_r$  can be embedded into the annulus  $A_r$  for  $r = 1, \dots, h$ .

An important deformation of the complex structure of  $C$  is the Dehn-twist: It corresponds to rotating one end of an annulus  $A_r$  by  $2\pi$  before regluing, and can be described by a change of the local coordinates used in the gluing construction. The coordinate  $q_r$  can not distinguish complex structures related by a Dehn twist in  $A_r$ . It is often useful to replace the coordinates

$q_r$  by logarithmic coordinates  $\tau_r$  such that  $q_r = e^{2\pi i \tau_r}$ . This corresponds to replacing the gluing identification (A.1) by its logarithm. In order to define the logarithms of the coordinates  $z_i$  used in (A.1), one needs to introduce branch cuts on the three-punctured spheres, an example being depicted in Figure 4.

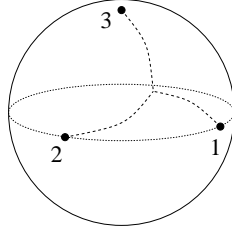


Figure 4: A sphere with three punctures, and a choice of branch cuts for the definition of the logarithms of local coordinates around the punctures.

By imposing the requirement that the branch cuts chosen on each three-punctured sphere glue to a connected three-valent graph  $\Gamma$  on  $C$ , one gets an unambiguous definition of the coordinates  $\tau_r$ . We see that the logarithmic versions of the gluing construction that define the coordinates  $\tau_r$  are parameterized by the pair of data  $\sigma = (\mathcal{C}_\sigma, \Gamma_\sigma)$ , where  $\mathcal{C}_\sigma$  is the cut system defined by the gluing construction, and  $\Gamma_\sigma$  is the three-valent graph specifying the choices of branch cuts. In order to have a handy terminology we will call the pair of data  $\sigma = (\mathcal{C}_\sigma, \Gamma_\sigma)$  a *pants decomposition*, and the three-valent graph  $\Gamma_\sigma$  will be called the Moore-Seiberg graph, or MS-graph associated to a pants decomposition  $\sigma$ . The construction outlined above gives a set of coordinates for the neighbourhood  $\mathcal{U}_\sigma$  of the boundary component of  $\mathcal{T}(C)$  corresponding to  $\sigma$ .

The gluing construction depends on the choices of coordinates around the punctures  $Q_i$ . There exists an ample supply of choices for the coordinates  $z_i$  such that the union of the neighbourhoods  $\mathcal{U}_\sigma$  produces a cover of  $\mathcal{M}(C)$  [HV]. For a fixed choice of these coordinates one produces families of Riemann surfaces fibred over the multi-discs  $\mathcal{U}_\sigma$  with coordinates  $q$ . Changing the coordinates  $z_i$  around  $Q_i$  produces a family of Riemann surfaces which is locally biholomorphic to the initial one [RS].

## A.2 The Moore-Seiberg groupoid

Let us note [MS, BK] that any two different pants decompositions  $\sigma_2, \sigma_1$  can be connected by a sequence of elementary moves localized in subsurfaces of  $C_{g,n}$  of type  $C_{0,3}$ ,  $C_{0,4}$  and  $C_{1,1}$ . The elementary moves are called the  $B$ ,  $F$ ,  $Z$  and  $S$ -moves, respectively. Graphical representations for the elementary moves  $F$ ,  $S$  and  $B$  are given in Figures 1, 2 and 3, respectively. The  $Z$ -move is just the change of distinguished boundary component in a three-punctured sphere.



One may formalize the resulting structure by introducing a two-dimensional CW complex  $\mathcal{M}(C)$  with set of vertices  $\mathcal{M}_0(C)$  given by the pants decompositions  $\sigma$ , and a set of edges  $\mathcal{M}_1(C)$  associated to the elementary moves. The Moore-Seiberg groupoid is defined to be the path groupoid of  $\mathcal{M}(C)$ . It can be described in terms of generators and relations, the generators being associated with the edges of  $\mathcal{M}(C)$ , and the relations associated with the faces of  $\mathcal{M}(C)$ . The classification of the relations was first presented in [MS], and rigorous mathematical proofs have been presented in [FG1, BK]. The relations are all represented by sequences of elementary moves localized in subsurfaces  $C_{g,n}$  with genus  $g = 0$  and  $n = 3, 4, 5$  punctures, as well as  $g = 1, n = 1, 2$ . Graphical representations of the relations can be found in [MS, FG1, BK].

### A.3 Uniformization

The classical uniformization theorem ensures existence and uniqueness of a hyperbolic metric, a metric of constant negative curvature, on a Riemann surface  $C$ . In a local chart with complex analytic coordinates  $y$  one may represent this metric in the form  $ds^2 = e^{2\varphi} dy d\bar{y}$ , with  $\varphi$  being a solution to the Liouville equation  $\partial\bar{\partial}\varphi = \mu e^{2\varphi} dy d\bar{y}$ .

The solutions to the Liouville equation may be parameterized by a function  $t(y)$  related to  $\varphi$  as

$$t := -(\partial_y \varphi)^2 + \partial_y^2 \varphi. \quad (\text{A.3})$$

$t(y)$  is holomorphic as a consequence of the Liouville equation. The solution to the Liouville equation can be reconstructed from  $t(y)$  by first finding the solutions to

$$(\partial_y^2 + t(y))\chi = 0. \quad (\text{A.4})$$

Picking two linearly independent solutions  $\chi_{\pm}$  of (A.4) with  $\chi'_+ \chi_- - \chi'_- \chi_+ = 1$  allows us to represent  $e^{2\varphi}$  as  $e^{2\varphi} = -(\chi_+ \bar{\chi}_- - \chi_- \bar{\chi}_+)^{-2}$ . The hyperbolic metric  $ds^2 = e^{2\varphi} dy d\bar{y}$  may then be written in terms of the quotient  $A(y) := \chi_+ / \chi_-$  as

$$ds^2 = e^{2\varphi} dy d\bar{y} = \frac{\partial A \bar{\partial} \bar{A}}{(\text{Im}(A))^2}. \quad (\text{A.5})$$

It follows that  $A(y)$  represents a conformal mapping from  $C$  to a domain  $\Omega$  in the upper half plane  $\mathbb{U}$  with its standard constant curvature metric. The monodromies of the solution  $\chi$  are represented on  $A(y)$  by Moebius transformations. These Moebius transformations describe the identifications of the boundaries of the simply-connected domain  $\Omega$  in  $\mathbb{U}$  which represents the image of  $C$  under  $A$ .  $C$  is therefore conformal to  $\mathbb{U}/\Gamma$ , where the Fuchsian group  $\Gamma$  is the monodromy group of the differential operator  $\partial_y^2 + t(y)$ .

## B. Moduli spaces of flat connections

In this appendix we shall review some of the basic definitions and results concerning the moduli spaces  $\mathcal{M}_{\text{flat}}(C)$ .

### B.1 Moduli of flat connections and character variety

We will consider flat  $\text{PSL}(2, \mathbb{C})$ -connections  $\nabla = d - A$  on Riemann surfaces  $C$ . Let  $\mathcal{M}_{\text{flat}}(C)$  be the moduli space of all such connections modulo gauge transformations.

Given a flat  $\text{PSL}(2, \mathbb{C})$ -connection  $\nabla = d - A$ , one may define its holonomy  $\rho(\gamma)$  along a closed loop  $\gamma$  as  $\rho(\gamma) = \mathcal{P} \exp(\int_{\gamma} A)$ . The assignment  $\gamma \mapsto \rho(\gamma)$  defines a representation of  $\pi_1(C)$  in  $\text{PSL}(2, \mathbb{C})$ . As any flat connection is locally gauge-equivalent to the trivial connection, one may characterize gauge-equivalence classes of flat connections by the corresponding representations  $\rho : \pi_1(C) \rightarrow \text{PSL}(2, \mathbb{C})$ . This allows us to identify the moduli space  $\mathcal{M}_{\text{flat}}(C)$  of flat  $\text{PSL}(2, \mathbb{C})$ -connections on  $C$  with the so-called character variety

$$\mathcal{M}_{\text{char}}(C) := \text{Hom}(\pi_1(C), \text{PSL}(2, \mathbb{C})) / \text{PSL}(2, \mathbb{C}). \quad (\text{B.1})$$

The moduli space  $\mathcal{M}_{\text{flat}}(C)$  has a natural real slice, the moduli space  $\mathcal{M}_{\text{flat}}^{\mathbb{R}}(C)$  of flat  $\text{PSL}(2, \mathbb{R})$ -connections.

### B.2 The Teichmüller component

There is a well-known relation between the Teichmüller space  $\mathcal{T}(C)$  and a connected component of the moduli space  $\mathcal{M}_{\text{flat}}^{\mathbb{R}}(C)$  of flat  $\text{PSL}(2, \mathbb{R})$ -connections on  $C$ . This component is called the Teichmüller component and will be denoted as  $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$ . The relation between  $\mathcal{T}(C)$  and  $\mathcal{M}_{\text{flat}}^0(C)$  may be described as follows. To a hyperbolic metric  $ds^2 = e^{2\varphi} dy d\bar{y}$  let us associate the connection  $\nabla = \nabla' + \nabla''$ ,

$$\nabla'' = \bar{\partial}, \quad \nabla' = \partial + M(y)dy, \quad M(y) = \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix}, \quad (\text{B.2})$$

with  $t$  constructed from  $\varphi(y, \bar{y})$  as in (A.3). This connection is flat since  $\partial_y \bar{\partial}_{\bar{y}} \varphi = \mu e^{2\varphi}$  implies  $\bar{\partial} t = 0$ . The Fuchsian group  $\Gamma$  characterizing the uniformization of  $C$  is nothing but the holonomy  $\rho$  of the connection  $\nabla$  defined in (B.2).

The Fuchsian groups  $\Gamma$  fill out the connected component  $\mathcal{M}_{\text{char}}^{\mathbb{R},0}(C) \simeq \mathcal{T}(C)$  in  $\mathcal{M}_{\text{flat}}^{\mathbb{R}}(C)$  called the Teichmüller component.

### B.3 Fock–Goncharov coordinates

Let  $\tau$  be a triangulation of the surface  $C$  such that all vertices coincide with marked points on  $C$ . An edge  $e$  of  $\tau$  separates two triangles defining a quadrilateral  $Q_e$  with corners being the marked points  $P_1, \dots, P_4$ . For a given local system  $(\mathcal{E}, \nabla)$ , let us choose four sections  $s_i, i = 1, 2, 3, 4$  that obey the condition  $\nabla s_i = 0$ , and are eigenvectors of the monodromy around  $P_i$ . Out of the sections  $s_i$  form [FG1, GMN2]

$$\mathcal{X}_e^\tau := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}, \quad (\text{B.3})$$

where all sections are evaluated at a common point  $P \in Q_e$ . It is not hard to see that  $\mathcal{X}_e^\tau$  does not depend on the choice of  $P$ .

There exists a simple description of the relations between the coordinates associated to different triangulations. If triangulation  $\tau_e$  is obtained from  $\tau$  by changing only the diagonal in the quadrangle containing  $e$ , we have

$$\mathcal{X}_{e'}^{\tau_e} = \begin{cases} \mathcal{X}_{e'}^\tau (1 + (\mathcal{X}_e^\tau)^{-\text{sgn}(n_{e'e})})^{-n_{e'e}} & \text{if } e' \neq e, \\ (\mathcal{X}_e^\tau)^{-1} & \text{if } e' = e. \end{cases} \quad (\text{B.4})$$

This reflects part of the structure of a cluster algebra that  $\mathcal{M}_{\text{flat}}(C)$  has.

### B.4 Trace functions

The trace functions

$$L_\gamma := \nu_\gamma \text{tr}(\rho(\gamma)), \quad (\text{B.5})$$

represent useful coordinate functions for  $\mathcal{M}_{\text{flat}}^{\mathbb{C}}(C)$ . The signs  $\nu_\gamma$  will be chosen such that the restriction to  $L_\gamma$  to the Teichmüller component  $\mathcal{M}_{\text{char}}^{\mathbb{R},0}(C)$  satisfies  $L_\gamma = 2 \cosh(l_\gamma/2) > 2$ , where  $l_\gamma$  is the length of the hyperbolic geodesic on  $\mathbb{U}/\Gamma$  isotopic to  $\gamma$ .

The coordinate functions  $L_\gamma$  generate the commutative algebra  $\mathcal{A}(C) \simeq \text{Fun}^{\text{alg}}(\mathcal{M}_{\text{flat}}(C))$  of functions on  $\mathcal{M}_{\text{flat}}(C)$ . The well-known relation  $\text{tr}(g)\text{tr}(h) = \text{tr}(gh) + \text{tr}(gh^{-1})$  valid for any pair of  $SL(2)$ -matrices  $g, h$  implies that the geodesic length functions satisfy the so-called skein relations,

$$L_{\gamma_1} L_{\gamma_2} = L_{S(\gamma_1, \gamma_2)}, \quad (\text{B.6})$$

where  $S(\gamma_1, \gamma_2)$  is the loop obtained from  $\gamma_1, \gamma_2$  by means of the smoothing operation, defined as follows. The application of  $S$  to a single intersection point of  $\gamma_1, \gamma_2$  is depicted in Figure 5 below. The general result is obtained by applying this rule at each intersection point, and summing the results.

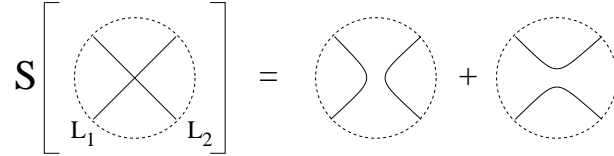


Figure 5: The symmetric smoothing operation

## B.5 Topological classification of closed loops

With the help of pants decompositions one may conveniently classify all non-selfintersecting closed loops on  $C$  up to homotopy. To a loop  $\gamma$  let us associate the collection of integers  $(r_e, s_e)$  associated to all edges  $e$  of  $\Gamma_\sigma$  which are defined as follows. Recall that there is a unique curve  $\gamma_e \in \mathcal{C}_\sigma$  that intersects a given edge  $e$  on  $\Gamma_\sigma$  exactly once, and which does not intersect any other edge. The integer  $r_e$  is defined as the number of intersections between  $\gamma$  and the curve  $\gamma_e$ . Having chosen an orientation for the edge  $e_r$  we will define  $s_e$  to be the intersection index between  $e$  and  $\gamma$ .

Dehn's theorem (see [DMO] for a nice discussion) ensures that the curve  $\gamma$  is up to homotopy uniquely classified by the collection of integers  $(r, s)$ , subject to the restrictions

$$\begin{aligned}
 & \text{(i)} \quad r_e \geq 0, \\
 & \text{(ii)} \quad \text{if } r_e = 0 \Rightarrow s_e \geq 0, \\
 & \text{(iii)} \quad r_{e_1} + r_{e_2} + r_{e_3} \in 2\mathbb{Z} \text{ whenever } \gamma_{e_1}, \gamma_{e_2}, \gamma_{e_3} \text{ bound the same trinion.}
 \end{aligned} \tag{B.7}$$

We will use the notation  $\gamma_{(r,s)}$  for the geodesic which has parameters  $(r, s) : e \mapsto (r_e, s_e)$ .

## B.6 Generators and relations

The pants decompositions allow us to describe  $\mathcal{A}(C)$  in terms of generators and relations. As set of generators for  $\mathcal{A}(C)$  one may take the functions  $L_{(r,s)} \equiv L_{\gamma_{(r,s)}}$ . The skein relations imply various relations among the  $L_{(r,s)}$ . It is not hard to see that these relations allow one to express arbitrary  $L_{(r,s)}$  in terms of a finite subset of the set of  $L_{(r,s)}$ .

Let us temporarily restrict attention to surfaces with genus zero and  $n = 4$  boundaries. The Moore-Seiberg graph  $\Gamma_\sigma$  will then have only one internal edge, allowing us to drop the index  $e$  labelling the edges. Let us introduce the geodesics  $\gamma_s = \gamma_{(1,0)}$ ,  $\gamma_t = \gamma_{(0,2)}$  and  $\gamma_u = \gamma_{(1,2)}$ . The geodesics  $\gamma_s$  and  $\gamma_t$  are depicted as red curves on the left and right half of Figure 1. We will denote  $L_k \equiv L_{\gamma_k}$ , where  $k \in \{s, t, u\}$ . The trace functions  $L_s, L_t$  and  $L_u$  generate  $\mathcal{A}(C)$ .

These coordinates are not independent, though. Further relations follow from the relations in  $\pi_1(C)$ . It can be shown (see e.g. [Go09] for a review) that the coordinate functions  $L_s, L_t$  and

$L_u$  satisfy an algebraic relation of the form

$$P(L_s, L_t, L_u) = 0. \quad (\text{B.8a})$$

The polynomial  $P$  in (B.8) is explicitly given as<sup>6</sup>

$$\begin{aligned} P(L_s, L_t, L_u) := & -L_s L_t L_u + L_s^2 + L_t^2 + L_u^2 \\ & + L_s(L_3 L_4 + L_1 L_2) + L_t(L_2 L_3 + L_1 L_4) + L_u(L_1 L_3 + L_2 L_4) \\ & - 4 + L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_1 L_2 L_3 L_4. \end{aligned} \quad (\text{B.8b})$$

In the expressions above we have denoted  $L_i := L_{\gamma_i}$ , where  $\gamma_i$ ,  $i = 1, 2, 3, 4$  represent the boundary components of  $C_{0,4}$ , labelled according to the convention defined in Figure 1.

## B.7 Trace functions in terms of Fock-Goncharov coordinates

Assume given a path  $\varpi_\gamma$  on the fat graph homotopic to a simple closed curve  $\gamma$  on  $C_{g,n}$ . Let the edges be labelled  $e_i$ ,  $i = 1, \dots, r$  according to the order in which they appear on  $\varpi_\gamma$ , and define  $\sigma_i$  to be 1 if the path turns left at the vertex that connects edges  $e_i$  and  $e_{i+1}$ , and to be equal to  $-1$  otherwise. Consider the following matrix,

$$X_\gamma = V^{\sigma_r} E(z_{e_r}) \cdots V^{\sigma_1} E(z_{e_1}), \quad (\text{B.9})$$

where  $z_e = \log X_e$ , and the matrices  $E(z)$  and  $V$  are defined respectively by

$$E(z) = \begin{pmatrix} 0 & +e^{+\frac{z}{2}} \\ -e^{-\frac{z}{2}} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{B.10})$$

Taking the trace of  $X_\gamma$  one gets the hyperbolic length of the closed geodesic isotopic to  $\gamma$  via [F97]

$$L_\gamma \equiv 2 \cosh(l_\gamma/2) = |\text{tr}(X_\gamma)|. \quad (\text{B.11})$$

We may observe that the classical expression for  $L_\gamma \equiv 2 \cosh \frac{1}{2} l_\gamma$  as given by formula B.11 is a linear combination of monomials in the variables  $u_e^{\pm 1} \equiv e^{\pm \frac{z_e}{2}}$  of the very particular form (5.4).

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<sup>6</sup>Comparing to [Go09] note that some signs were absorbed by a suitable choice of the signs  $\nu_\gamma$  in (B.5).

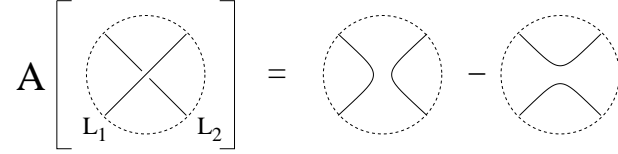


Figure 6: The anti-symmetric smoothing operation

### B.8 Fenchel-Nielsen coordinates for $\mathcal{M}_{\text{flat}}^{\mathbb{R},0}(C)$

One may express  $L_s$ ,  $L_t$  and  $L_u$  in terms of the Fenchel-Nielsen coordinates  $l$  and  $k$  [Ok, Go09]. Explicit expressions are for  $C_{0,4}$ ,

$$L_s = 2 \cosh(l/2), \quad (\text{B.12a})$$

$$L_t((L_s)^2 - 4) = 2(L_2L_3 + L_1L_4) + L_s(L_1L_3 + L_2L_4) + 2 \cosh(k) \sqrt{c_{12}(L_s)c_{34}(L_s)}, \quad (\text{B.12b})$$

$$L_u((L_s)^2 - 4) = L_s(L_2L_3 + L_1L_4) + 2(L_1L_3 + L_2L_4) + 2 \cosh((2k - l)/2) \sqrt{c_{12}(L_s)c_{34}(L_s)}, \quad (\text{B.12c})$$

where  $L_i = 2 \cosh \frac{l_i}{2}$ , and  $c_{ij}(L_s)$  is defined as

$$c_{ij}(L_s) = L_s^2 + L_i^2 + L_j^2 + L_s L_i L_j - 4. \quad (\text{B.13})$$

These expressions ensure that the algebraic relations  $P_e(L_s, L_t, L_u) = 0$  are satisfied. By complexifying  $(l, k)$  one gets (local) coordinates for  $\mathcal{M}_{\text{flat}}^{\mathbb{C}}(C)$  [NRS].

### B.9 Poisson structure

There is also a natural Poisson bracket on  $\mathcal{A}(C)$  [Go86], defined such that

$$\{L_{\gamma_1}, L_{\gamma_2}\} = L_{A(\gamma_1, \gamma_2)}, \quad (\text{B.14})$$

where  $A(\gamma_1, \gamma_2)$  is the loop obtained from  $\gamma_1, \gamma_2$  by means of the anti-symmetric smoothing operation, defined as above, but replacing the rule depicted in Figure 5 by the one depicted in Figure 6. This Poisson structure coincides with the Poisson structure coming from the natural symplectic structure on  $\mathcal{M}_{\text{flat}}(C)$  which was introduced by Atiyah and Bott.

The resulting expression for the Poisson bracket  $\{L_s, L_t\}$  can be written elegantly in the form

$$\{L_s, L_t\} = \frac{\partial}{\partial L_u} P(L_s, L_t, L_u). \quad (\text{B.15})$$

It is remarkable that the same polynomial appears both in (B.8) and in (B.15), which indicates that the symplectic structure on  $\mathcal{M}_{\text{flat}}$  is compatible with its structure as algebraic variety.

The Fenchel-Nielsen coordinates are known to be Darboux-coordinates for  $\mathcal{M}_{\text{flat}}(C)$ , having the Poisson bracket

$$\{l, k\} = 2. \quad (\text{B.16})$$

The Poisson structure is also rather simple in terms of the Fock-Goncharov coordinates,

$$\{\mathcal{X}_e^\tau, \mathcal{X}_{e'}^\tau\} = n_{e,e'} \mathcal{X}_{e'}^\tau \mathcal{X}_e^\tau, \quad (\text{B.17})$$

where  $n_{e,e'}$  is the number of faces  $e$  and  $e'$  have in common, counted with a sign.

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